Describing, through Maple, GeoGebra Discovery's new features: proof certificates, complexity estimation

Zoltán Kovács. The Private University College of Education of the Diocese of Linz, Austria. zoltan.kovacs@ph-linz.at

Tomás Recio. Universidad Antonio de Nebrija, Madrid, Spain. trecio@nebrija.es

M. Pilar Vélez. Universidad Antonio de Nebrija, Madrid, Spain. pvelez@nebrija.es

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The idea of ranking geometry theorems

Wos Research Problem 31 (1988)

What properties can be identified to permit an automated reasoning program to find new and interesting theorems, as opposed to proving conjectured theorems?

Also studied by

- Colton and Bundy (2000)
- Puzis, Gao and Sutcliffe (2006)
- Gao, Goto and Cheng (2015)
- Gao, Li and Cheng (2019)
- Quaresma and Graziani (2023)

We present a new GeoGebra Discovery command, ShowProof, which provides

- a certified step-by-step algebraic proof, and
- **complexity grade** of planar geometry theorems.

The output is available in multiple languages (English, Arabic, French , German, Hebrew, Hungarian, Italian, Spanish and Turkish) and can be exported as plain text to a

- Mathematica,
- Maple or
- Giac

format for further analysis.

Why are we interested in proof certificates, complexity degree computations?

a) improving the performance of GeoGebra regarding the grade of the addressed problems,

b) or by selecting, in the *Discover* or *Automated Geometer* command output, to exhibit only interesting theorems, avoiding obvious, trivial ones, etc.

c) adapting GeoGebra's reasoning tools to the needs of students with special needs,

d) analyzing the potential role of GeoGebra concerning geometric problems posed in mathematical contests, such as the mathematical Olympiads, etc.

e) Providing benchmarks for Data Science

Hongbiao Gao, Yuichi Goto, and Jingde Cheng. A Set of Metrics for Measuring Interestingness of Theorems in Automated Theorem Finding by Forward Reasoning: A Case Study in NBG Set Theory. Springer International Publishing Switzerland 2015. X. He et al. (Eds.): IScIDE (Intelligence Science and Big Data Engineering. Big Data and Machine Learning Techniques) 2015, Part II, LNCS 9243, pp. 508–517, 2015. DOI: 10.1007/978-3-319-23862-3 50

H. Gao, J. Li and J. Cheng, "Measuring Interestingness of Theorems in Automated Theorem Finding by Forward Reasoning: A Case Study in Tarski's Geometry," *2018 IEEE SmartWorld, Ubiquitous Intelligence & Computing, Advanced & Trusted Computing, Scalable Computing & Communications, Cloud & Big Data Computing, Internet of People and Smart City Innovation (SmartWorld/SCALCOM/UIC/ATC/CBDCom/IOP/SCI)*, 2018, pp. 168-173, doi: 10.1109/SmartWorld.2018.00064.

Gao, H.; Li, J.; Cheng J. Measuring Interestingness of Theorems in Automated Theorem Finding by Forward Reasoning Based on Strong Relevant Logic. In: 2019 IEEE International Conference on Energy Internet (ICEI), pp. 356–361, doi:10.1109/ICEI.2019.00069.

f) Collaboration with ChatGPT, AI

F. Botana, T. Recio, M.P. Vélez: On using GeoGebra and ChatGPT for geometric discovery (Computers -3105885)

https://www.mdpi.com/2079-3197/12/2/30

Even if the question (defining “interesting results” for automated methods to take into consideration) goes back to 1988, 35 years ago, is now urgent and new because

--other colleagues are approaching the same issue, with a different perspective

--we have a tool that produces automatically geometric statements, it is necessary to have a tool to select them

--colleagues are approaching the automated production of, say, olympiad problems, so “measuring” its difficulty could be relevant

--likewise, we could apply our approach to assessing teachers on the selection of mathematical problems adapted (their difficulty) to students with special needs.

Ranking geometry theorems

Our proposal – via algebraic geometry

Difficulty:

Comparison of
the expression of thesis
in terms of
the expression of the hypotheses

Input and output

Given:

- Free points F1, F2, . . ., Fm and constructed (dependent) points D1, D2, . . ., Dn via construction steps (hypotheses) h1, h2, . . .(as multivariate polynomials over Q).
- A geometric statement (thesis) t (as a multivar. poly over Q).
- Expected: to get a machine generated readable proof for h1, h2, . . . ⇒t.
- Output: If required, some additional hypotheses (non-degeneracy conditions) hr are detected, and it is certified with polynomials f1, f2, . . . , f_{r+1} that

$$f1 \cdot h1 + \ldots + fr \cdot hr + f\_\{r+1\}(t \cdot z - 1) = 1$$

where fi are polys in the slack variable z and in the variables appearing in the hypotheses eqs. (Proof by contradiction: 0= 1.)

Triangle A (a1,a2), B(b1,b2), C(c1,c2)
Hypothesis: Right angle at A
AB perpendicular to AC: (a1-b1)*(a1-c1)+(a2-b2)*(a2-c2)

>Expand:

a1^2 - a1*b1 - a1*c1 + a2^2 - a2*b2 - a2*c2 + b1*c1 + b2*c2

Thesis: Pythagoras' Th.
(a1-b1)^2+(a2-b2)^2+ (a1-c1)^2+(a2-c2)^2=(c1-b1)^2+(c2-b2)^2

> Expand: (a1-b1)^2+(a2-b2)^2+ (a1-c1)^2+(a2-c2)^2-((c1-b1)^2+(c2-b2)^2)

2*a1^2 – 2*a1*b1 - 2*a1*c1 + 2*a2^2 - 2*a2*b2 - 2*a2*c2 + 2*b1*c1 + 2*b2*c2

Thesis= 2* Hypothesis
(complexity 0)

Proof by contradiction:

$$\text{Hypotheses} + \text{Negation of Thesis} = 1$$

$$1 = z^* \, 2^* \text{hypothesis} - (\text{thesis}^* z - 1)$$

Here, complexity 1.

A = (-7.97, 1.92)

B = (-4.03, -1.38)

f : PerpendicularBisector(A, B)

→ -3.94x + 3.3y = 24.53

C = Point(f)

→ (-4.32, 2.28)

g = Segment(A, C)

→ 3.67

h = Segment(B, C)

→ 3.67

Input...

1 Let A, B be arbitrary points.

2 Let f be the perpendicular bisector of AB.

3 Let C be a point on f.

4 Let g be the segment A, C.

5 Let h be the segment B, C.

6 Prove that g ≟ h.

7 The statement is always true.

8 We prove this by contradiction.

9 Let free point A be denoted by (v1,v2).

10 Let free point B be denoted by (v3,v4).

11 Considering definition f = PerpendicularBisector[A, B]:

12 e1 : $2\,v5 - v3 - v1 = 0$
→ **e1 : $-v1 - v3 + 2\,v5 = 0$**

13 e2 : $2\,v6 - v4 - v2 = 0$
→ **e2 : $-v2 - v4 + 2\,v6 = 0$**

14 e3 : $v8 - v6 - v5 + v1 = 0$
→ **e3 : $v1 - v5 - v6 + v8 = 0$**

15 e4 : $v7 + v6 - v5 - v2 = 0$
→ **e4 : $-v2 - v5 + v6 + v7 = 0$**

16 Considering definition C = Point[f]:

17 Let dependent point C be denoted by (v9,v10).

18 e5 : $-v9\,v8 + v10\,v7 + v9\,v6 - v7\,v6 - v10\,v5 + v8\,v5 = 0$
→ **e5 : $-v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9 = 0$**

19 Thesis reductio ad absurdum (denied statement):

20 e6 : $-1 + 2\,v11\,v10\,v4 - v11\,v4^2 + 2\,v11\,v9\,v3 - v11\,v3^2 - 2\,v11\,v10\,v2 + v11\,v2^2 - 2\,v11\,v9\,v1 + v11\,v1^2 = 0$
→ **e6 : $v11\,v2^2 - v11\,v3^2 - v11\,v4^2 + v1^2\,v11 - 2\,v1\,v11\,v9 - 2\,v10\,v11\,v2 + 2\,v10\,v11\,v4 + 2\,v11\,v3\,v9 - 1 = 0$**

21 Without loss of generality, some coordinates can be fixed:

22 $\{v1 = 0, v2 = 0\}$
→ **$\{v1 = 0, v2 = 0\}$**

23 All hypotheses and the negated thesis after substitutions:

24 s1 : $2\,v5 - v3 = 0$
→ **s1 : $-v3 + 2\,v5 = 0$**

25 s2 : $2\,v6 - v4 = 0$
→ **s2 : $-v4 + 2\,v6 = 0$**

26 s3 : $v8 - v6 - v5 = 0$
→ **s3 : $-v5 - v6 + v8 = 0$**

27 s4 : $v7 + v6 - v5 = 0$
→ **s4 : $-v5 + v6 + v7 = 0$**

28 s5 : $-v9\,v8 + v10\,v7 + v9\,v6 - v7\,v6 - v10\,v5 + v8\,v5 = 0$
→ **s5 : $-v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9 = 0$**

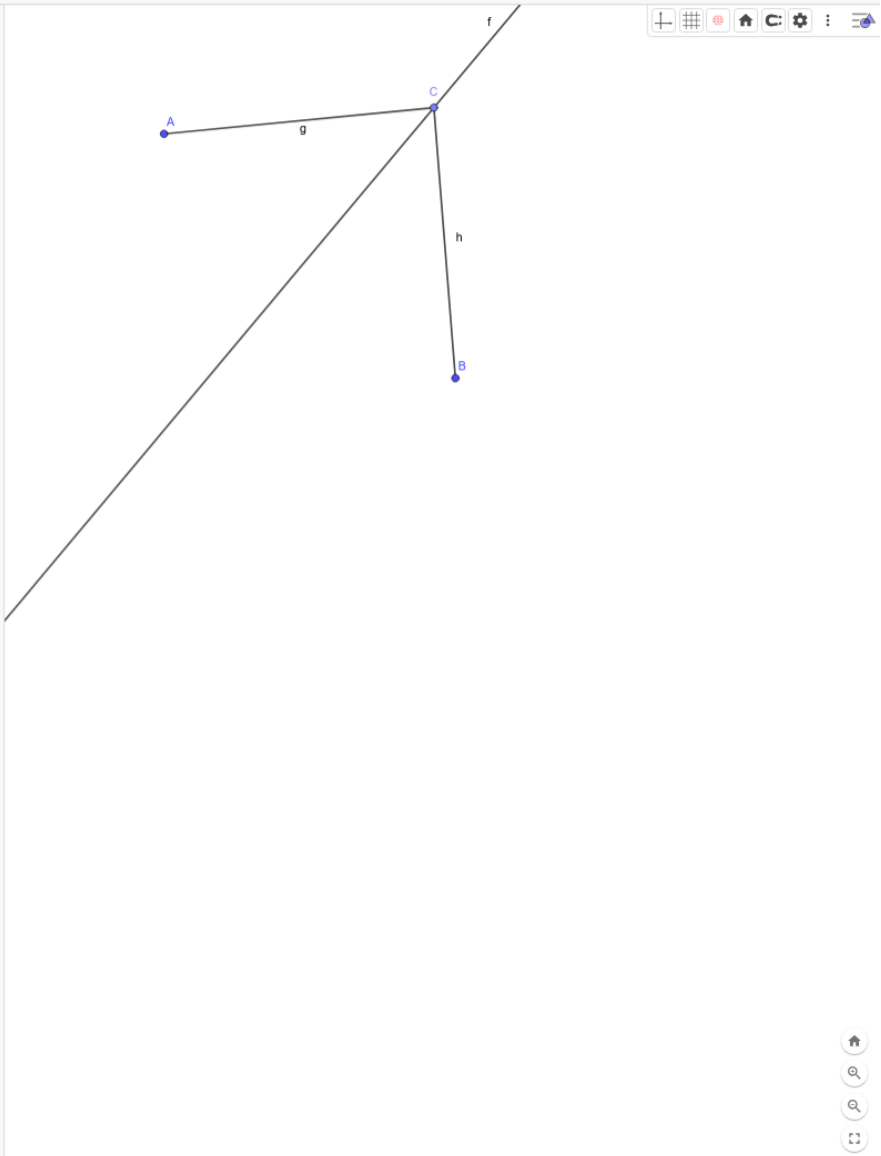
29 s6 : $-1 + 2\,v11\,v10\,v4 - v11\,v4^2 + 2\,v11\,v9\,v3 - v11\,v3^2 = 0$
→ **s6 : $-v11\,v3^2 - v11\,v4^2 + 2\,v10\,v11\,v4 + 2\,v11\,v3\,v9 - 1 = 0$**

30 Now we consider the following expression:

31 $s1\,(v11\,v3 - v11\,v4 + 2\,v11\,v8 - 2\,v11\,v9) + s2\,(-2\,v10\,v11 + v11\,v3 + v11\,v4 - 2\,v11\,v7) + s3\,(2\,v11\,v3 - 4\,v11\,v9) + s4\,(4\,v10\,v11 - 2\,v11\,v4) + s5\,(-4\,v11) + s6\,(-1)$
→ **$1 = 0$**

32 **Contradiction! This proves the original statement.**

33 The statement has a difficulty of degree 2.



```
> restart:with(PolynomialIdeals):
>
> tesis:=v1^2- 2*v1*v9 - 2*v10*v2 + 2*v10*v4 + v2^2 - v3^2 + 2*v3*v9 - v4^2;-v1-v3+2*v5,-v2-
  v4+2*v6,v1-v5-v6+v8,-v2-v5+v6+v7, -v10*v5+v10*v7+v5*v8-v6*v7+v6*v9-v8*v9,v11*v2^2-v11*v3^2-
  v11*v4^2+v11*v1^2-2*v11*v1*v9-2*v10*v11*v2+2*v10*v11*v4+2*v11*v3*v9-1;
```

$$tesis := v1^2 - 2\,v1\,v9 - 2\,v10\,v2 + 2\,v10\,v4 + v2^2 - v3^2 + 2\,v3\,v9 - v4^2$$

$$-v1 - v3 + 2\,v5,\ -v2 - v4 + 2\,v6,\ v1 - v5 - v6 + v8,\ -v2 - v5 + v6 + v7,\ -v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9,\ v11\,v1^2 - 2\,v11\,v1\,v9 - 2\,v10\,v11\,v2 + 2\,v10\,v11\,v4 + v11\,v2^2 - v11\,v3^2 + 2\,v11\,v3\,v9 - v11\,v4^2 - 1 \quad (1.1)$$

```
> tesis*v11-1;
```

$$\left(v1^2 - 2\,v1\,v9 - 2\,v10\,v2 + 2\,v10\,v4 + v2^2 - v3^2 + 2\,v3\,v9 - v4^2\right) v11 - 1 \quad (1.2)$$

```
> simplify(%-(v11*v2^2-v11*v3^2-v11*v4^2+v11*v1^2-2*v11*v1*v9-2*v10*v11*v2+2*v10*v11*v4+2*v11*
  v3*v9-1));
```

$$0 \quad (1.3)$$

```
> hipo:=<-v1-v3+2*v5,-v2-v4+2*v6,v1-v5-v6+v8,-v2-v5+v6+v7, -v10*v5+v10*v7+v5*v8-v6*v7+v6*v9-v8*
  v9>;
```

$$hipo := \langle -v1 - v3 + 2\,v5,\ -v2 - v4 + 2\,v6,\ v1 - v5 - v6 + v8,\ -v2 - v5 + v6 + v7,\ -v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9 \rangle \quad (1.4)$$

```
> tesis in hipo;
```

$$true \quad (1.5)$$

```
> 1 in <-v1 - v3 + 2*v5, -v2 - v4 + 2*v6, v1 - v5 - v6 + v8, -v2 - v5 + v6 + v7, -v10*v5 + v10*
  v7 + v5*v8 - v6*v7 + v6*v9 - v8*v9, v10*v5+v10*v7+v5*v8-v6*v7+v6*v9-v8*v9,v11*v2^2-v11*v3^2-
  v11*v4^2+v11*v1^2-2*v11*v1*v9-2*v10*v11*v2+2*v10*v11*v4+2*v11*v3*v9-1>;
```

$$true \quad (1.6)$$

```
> HilbertDimension(hipo);
```

$$5 \quad (1.7)$$

```
> EliminationIdeal(hipo,{v1,v2,v3,v4,v10});EliminationIdeal(hipo,{v1,v2,v3,v4,v9});
```

$$\langle 0 \rangle$$

$$\langle 0 \rangle \quad (1.8)$$

```
> with(Groebner):  F:=[-v1-v3+2*v5,-v2-v4+2*v6,v1-v5-v6+v8,-v2-v5+v6+v7, -v10*v5+v10*v7+v5*v8-
  v6*v7+v6*v9-v8*v9]; nops(F);G, C := Basis(F, tdeg(v1,v2,v3,v4,v5,v6,v7,v8,v9,v10), output =
  extended);nops(C);
```

$$F := [-v1 - v3 + 2\,v5,\ -v2 - v4 + 2\,v6,\ v1 - v5 - v6 + v8,\ -v2 - v5 + v6 + v7,\ -v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9]$$

$$5$$

$$G, C := [-v5 - v6 + v7 + v4,\ -v5 + v6 - v8 + v3,\ v2 + v5 - v6 - v7,\ v1 - v5 - v6 + v8,\ v10\,v5 - v10\,v7 - v5\,v8 + v6\,v7 - v6\,v9 + v8\,v9],\ [[0, -1, 0, 1, 0], [-1, 0, -1, 0, 0], [0, 0, 0, -1, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, -1]]$$

$$5 \tag{1.9}$$

```
> NormalForm(tesis, G, tdeg(v1,v2,v3,v4,v5,v6,v7,v8,v9,v10), 'Q');
```

$$0 \tag{1.10}$$

```
> Q;nops(Q);
```

$$[-v4 - v5 - v6 + v7 + 2\,v10,\ -v3 - v5 + v6 - v8 + 2\,v9,\ v2 - v5 + v6 + v7 - 2\,v10,\ v1 + v5 + v6 - v8 - 2\,v9,\ 4]$$

$$5 \tag{1.11}$$

```
> C[1];
```

$$[0, -1, 0, 1, 0] \tag{1.12}$$

```
> C[2];
```

$$[-1, 0, -1, 0, 0] \tag{1.13}$$

```
> C[3];
```

$$[0, 0, 0, -1, 0] \tag{1.14}$$

```
> C[4];
```

$$[0, 0, 1, 0, 0] \tag{1.15}$$

```
> C[5];
```

$$[0, 0, 0, 0, -1] \tag{1.16}$$

```
>
```

```
> add(C[1][i]*F[i],i=1..nops(F))*Q[1]+add(C[2][i]*F[i],
  i=1..nops(F))*Q[2]+add(C[3][i]*F[i],i=1..nops(F))*Q[3]
  +add(C[4][i]*F[i],i=1..nops(F))*Q[4]+add(C[5][i]*F[i],
  i=1..nops(F))*Q[5];simplify(tesis-%);
```

$$\begin{aligned}&(-v5-v6+v7+v4)\,(-v4-v5-v6+v7+2\,v10)+(-v5+v6-v8\\&\quad+v3)\,(-v3-v5+v6-v8+2\,v9)+(v2+v5-v6-v7)\,(v2-v5\\&\quad+v6+v7-2\,v10)+(v1-v5-v6+v8)\,(v1+v5+v6-v8-2\,v9)\\&\quad+4\,v10\,v5-4\,v10\,v7-4\,v5\,v8+4\,v6\,v7-4\,v6\,v9+4\,v8\,v9\end{aligned}$$

$$0 \qquad \textbf{(1.17)}$$

Contradiction, complexity 2.

```
> with(Groebner):  FF:=[-v1 - v3 + 2*v5, -v2 - v4 + 2*v6, v1 - v5 - v6 + v8, -v2 - v5 + v6 +
  v7, -v10*v5 + v10*v7 + v5*v8 - v6*v7 + v6*v9 - v8*v9, v10*v5+v10*v7+v5*v8-v6*v7+v6*v9-v8*v9,
  v11*v2^2-v11*v3^2-v11*v4^2+v11*v1^2-2*v11*v1*v9-2*v10*v11*v2+2*v10*v11*v4+2*v11*v3*v9-1];
  nops(FF);GG, CC := Basis(FF, tdeg(v1,v2,v3,v4,v5,v6,v7,v8,v9,v10,v11), output = extended);
  nops(CC);
```

$$FF := \left[-v1 - v3 + 2\,v5,\ -v2 - v4 + 2\,v6,\ v1 - v5 - v6 + v8,\ -v2 - v5 + v6 + v7,\ -v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9,\right.$$
$$v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9,\ v11\,v1^2 - 2\,v11\,v1\,v9 - 2\,v10\,v11\,v2 + 2\,v10\,v11\,v4 + v11\,v2^2 - v11\,v3^2$$
$$\left. + 2\,v11\,v3\,v9 - v11\,v4^2 - 1\right]$$

$$7$$

$$GG,\ CC := [1],\ [[v11\,v3 + v11\,v5 - v6\,v11 + v8\,v11 - 2\,v11\,v9,\ -2\,v11\,v10 + v11\,v4 + v11\,v5 + v6\,v11 - v7\,v11,\ v1\,v11 + v11\,v3$$
$$+ 2\,v11\,v5 - 4\,v11\,v9,\ 4\,v11\,v10 - v2\,v11 - v11\,v4 - 2\,v6\,v11,\ -4\,v11,\ 0,\ -1]]$$

$$1 \qquad \textbf{(2.1)}$$

```
> NormalForm(1, GG, tdeg(v1,v2,v3,v4,v5,v6,v7,v8,v9,v10,v11), 'QQ');
```

$$0 \qquad \textbf{(2.2)}$$

```
> QQ;nops(QQ);
```

$$[1]$$

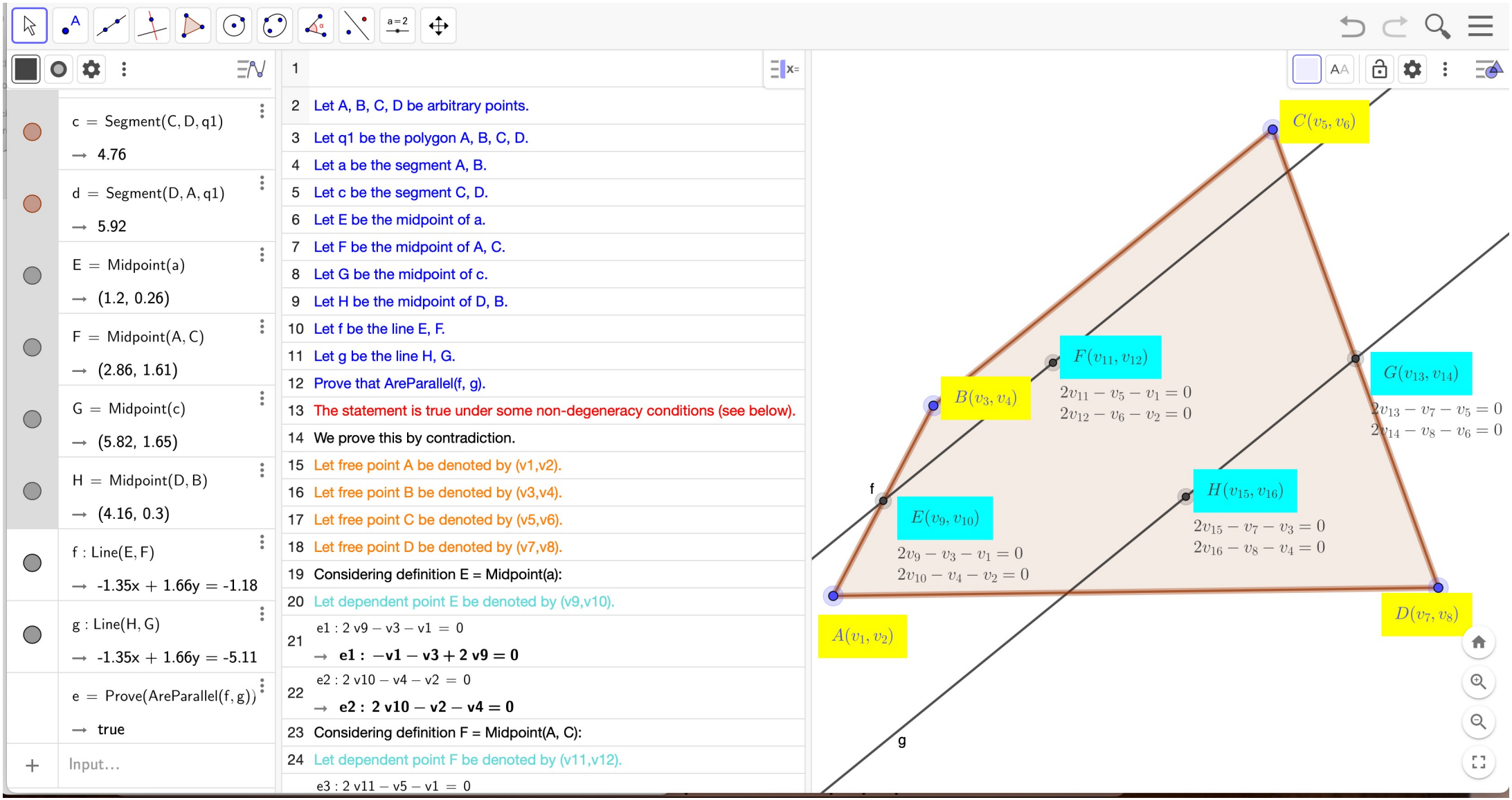
$$1 \qquad \textbf{(2.3)}$$

```
> add(CC[1][i]*FF[i],i=1..nops(FF));simplify(1-%);
```

$$(v11\,v3 + v11\,v5 - v6\,v11 + v8\,v11 - 2\,v11\,v9)\,(-v1 - v3 + 2\,v5) + (-2\,v11\,v10 + v11\,v4 + v11\,v5 + v6\,v11 - v7\,v11)\,(-v2 - v4$$
$$+ 2\,v6) + (v1\,v11 + v11\,v3 + 2\,v11\,v5 - 4\,v11\,v9)\,(v1 - v5 - v6 + v8) + (4\,v11\,v10 - v2\,v11 - v11\,v4 - 2\,v6\,v11)\,(-v2 - v5$$
$$+ v6 + v7) - 4\,v11\,(-v10\,v5 + v10\,v7 + v5\,v8 - v6\,v7 + v6\,v9 - v8\,v9) - v11\,v1^2 + 2\,v11\,v1\,v9 + 2\,v10\,v11\,v2 - 2\,v10\,v11\,v4$$
$$- v11\,v2^2 + v11\,v3^2 - 2\,v11\,v3\,v9 + v11\,v4^2 + 1$$

$$0 \qquad \textbf{(2.4)}$$

```
>
```


c = Segment(C, D, q1)
→ 4.76
d = Segment(D, A, q1)
→ 5.92
E = Midpoint(a)
→ (1.2, 0.26)
F = Midpoint(A, C)
→ (2.86, 1.61)
G = Midpoint(c)
→ (5.82, 1.65)
H = Midpoint(D, B)
→ (4.16, 0.3)
f : Line(E, F)
→ -1.35x + 1.66y = -1.18
g : Line(H, G)
→ -1.35x + 1.66y = -5.11
e = Prove(AreParallel(f, g))
→ true
Input...
1
2 Let A, B, C, D be arbitrary points.
3 Let q1 be the polygon A, B, C, D.
4 Let a be the segment A, B.
5 Let c be the segment C, D.
6 Let E be the midpoint of a.
7 Let F be the midpoint of A, C.
8 Let G be the midpoint of c.
9 Let H be the midpoint of D, B.
10 Let f be the line E, F.
11 Let g be the line H, G.
12 Prove that AreParallel(f, g).
13 The statement is true under some non-degeneracy conditions (see below).
14 We prove this by contradiction.
15 Let free point A be denoted by (v1,v2).
16 Let free point B be denoted by (v3,v4).
17 Let free point C be denoted by (v5,v6).
18 Let free point D be denoted by (v7,v8).
19 Considering definition E = Midpoint(a):
20 Let dependent point E be denoted by (v9,v10).
21 e1 : 2 v9 − v3 − v1 = 0
→ e1 : −v1 − v3 + 2 v9 = 0
22 e2 : 2 v10 − v4 − v2 = 0
→ e2 : 2 v10 − v2 − v4 = 0
23 Considering definition F = Midpoint(A, C):
24 Let dependent point F be denoted by (v11,v12).
e3 : 2 v11 − v5 − v1 = 0
$C(v_5, v_6)$
$F(v_{11}, v_{12})$
$2v_{11} - v_5 - v_1 = 0$
$2v_{12} - v_6 - v_2 = 0$
$G(v_{13}, v_{14})$
$2v_{13} - v_7 - v_5 = 0$
$2v_{14} - v_8 - v_6 = 0$
$B(v_3, v_4)$
f
$H(v_{15}, v_{16})$
$2v_{15} - v_7 - v_3 = 0$
$2v_{16} - v_8 - v_4 = 0$
$E(v_9, v_{10})$
$2v_9 - v_3 - v_1 = 0$
$2v_{10} - v_4 - v_2 = 0$
$D(v_7, v_8)$
$A(v_1, v_2)$
g

A = (0.71, -0.67)

B = (1.69, 1.19)

C = (5.01, 3.89)

D = (6.63, -0.59)

q1 = Polygon(A, B, C, D)
→ 15.09

a = Segment(A, B, q1)
→ 2.1

b = Segment(B, C, q1)
→ 4.28

c = Segment(C, D, q1)
→ 4.76

d = Segment(D, A, q1)
→ 5.92

E = Midpoint(a)
→ (1.2, 0.26)

F = Midpoint(A, C)
→ (2.86, 1.61)

41 Without loss of generality, some coordinates can be fixed:

42 $\{v1 = 0, v2 = 0, v3 = 0, v4 = 1\}$
→ $\mathbf{\{v1 = 0, v2 = 0, v3 = 0, v4 = 1\}}$

43 All hypotheses and the negated thesis after substitutions:

44 $s1 : 2\ v9 = 0$
→ $\mathbf{s1 : 2\ v9 = 0}$

45 $s2 : -1 + 2\ v10 = 0$
→ $\mathbf{s2 : 2\ v10 - 1 = 0}$

46 $s3 : 2\ v11 - v5 = 0$
→ $\mathbf{s3 : 2\ v11 - v5 = 0}$

47 $s4 : 2\ v12 - v6 = 0$
→ $\mathbf{s4 : 2\ v12 - v6 = 0}$

48 $s5 : 2\ v13 - v7 - v5 = 0$
→ $\mathbf{s5 : 2\ v13 - v5 - v7 = 0}$

49 $s6 : 2\ v14 - v8 - v6 = 0$
→ $\mathbf{s6 : 2\ v14 - v6 - v8 = 0}$

50 $s7 : 2\ v15 - v7 = 0$
→ $\mathbf{s7 : 2\ v15 - v7 = 0}$

51 $s8 : -1 + 2\ v16 - v8 = 0$
→ $\mathbf{s8 : 2\ v16 - v8 - 1 = 0}$

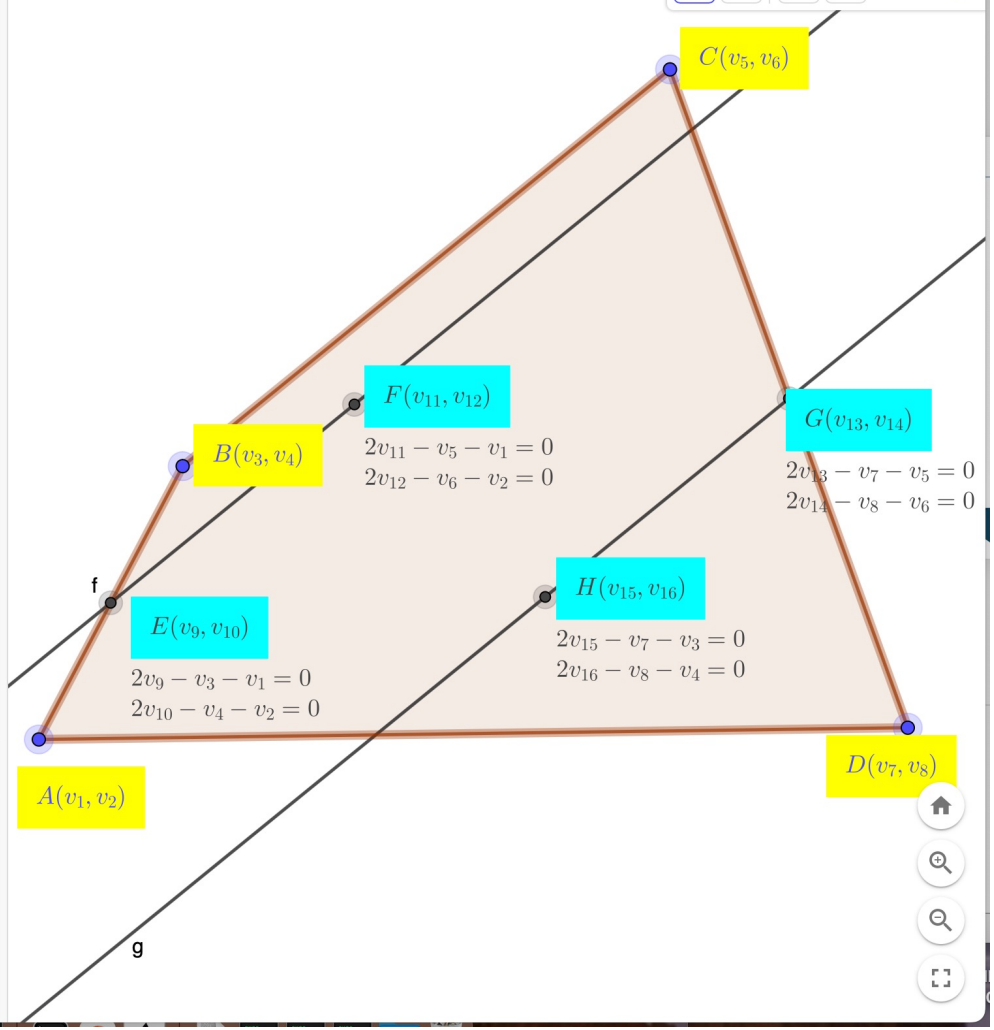
52 $s9 : -1 + v17\ v15\ v12 - v17\ v13\ v12 - v17\ v16\ v11 + v17\ v14\ v11 - v17\ v15\ v10 +$
→ $\mathbf{s9 : v10\ v13\ v17 - v10\ v15\ v17 + v11\ v14\ v17 - v11\ v16\ v17 - v12\ v}$

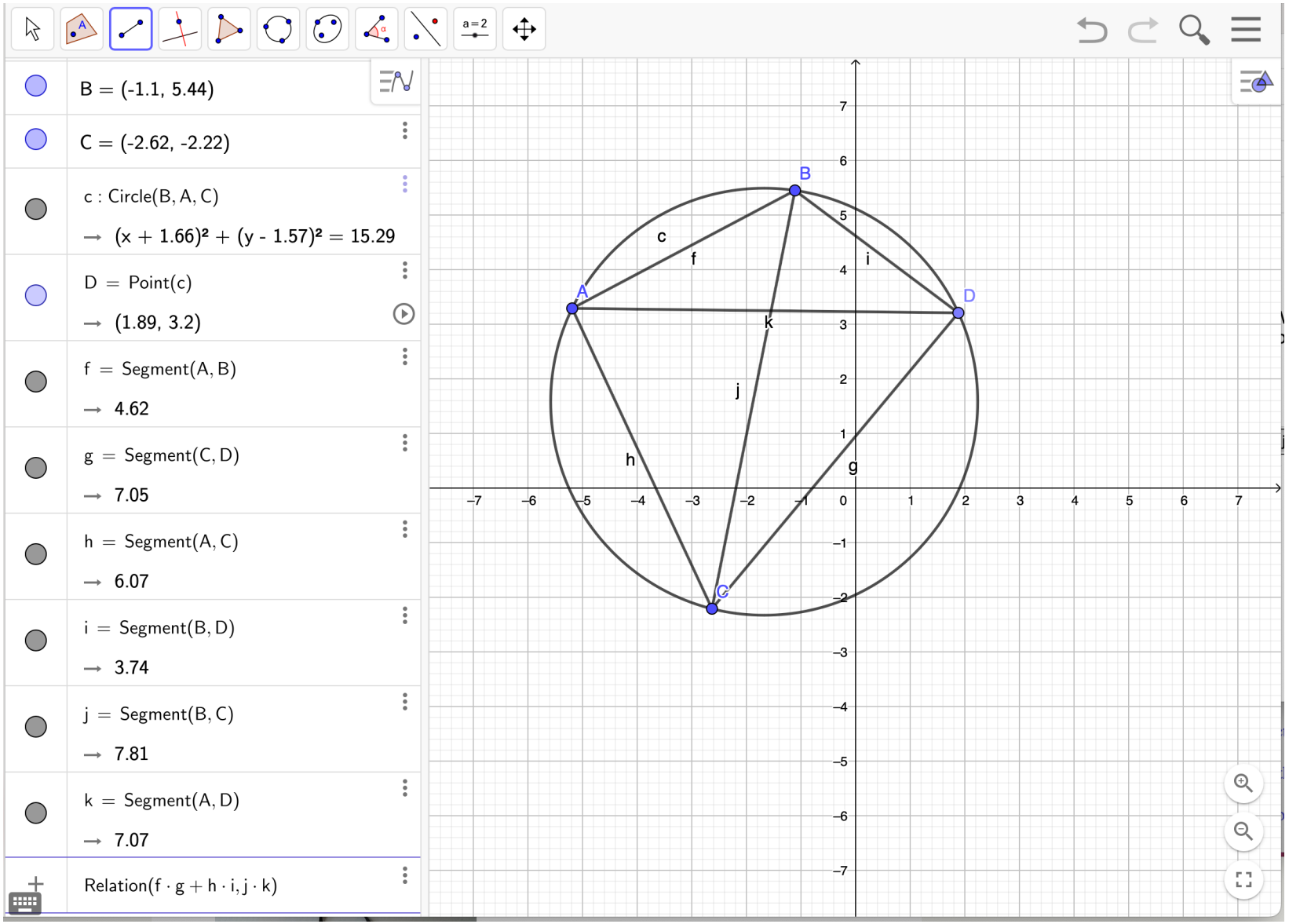
53 Now we consider the following equation:

54 $s1\left(\frac{-1}{2}\ v14\ v17 + \frac{1}{2}\ v16\ v17\right) + s2\left(\frac{1}{2}\ v13\ v17 - \frac{1}{2}\ v15\ v17\right) + s3\left(\frac{1}{2}\ v14\ v17 -\right.$
→ $\mathbf{1 = 0}$

55 **Contradiction! This proves the original statement.**

56 The statement has a difficulty of degree 2.




B = (-1.1, 5.44)
C = (-2.62, -2.22)
c : Circle(B, A, C)
→ (x + 1.66)² + (y - 1.57)² = 15.29
D = Point(c)
→ (1.89, 3.2)
f = Segment(A, B)
→ 4.62
g = Segment(C, D)
→ 7.05
h = Segment(A, C)
→ 6.07
i = Segment(B, D)
→ 3.74
j = Segment(B, C)
→ 7.81
k = Segment(A, D)
→ 7.07
Relation(f · g + h · i, j · k)
A
B
C
D
c
f
i
k
j
h
g

ptolomeo.ggb

Algebra

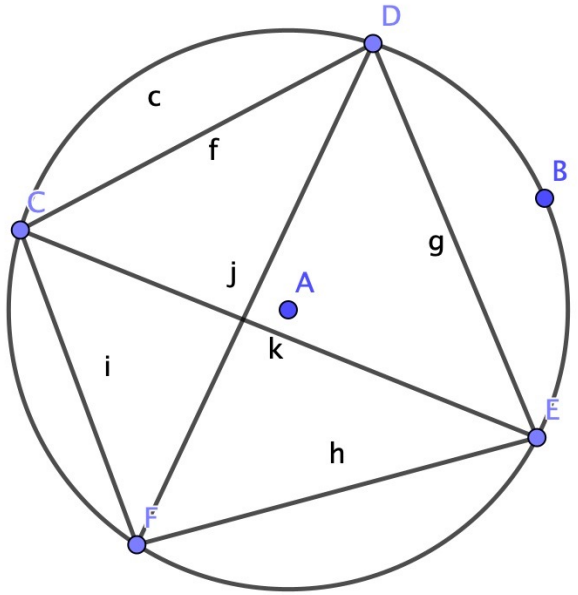
- A = (0, 2)
- B = (2.94, 3.28)
- c: $x^2 + (y - 2)^2 = 10.2$
- C = (-3.07, 2.92)
- D = (0.97, 5.06)
- E = (2.85, 0.53)
- F = (-1.74, -0.7)
- f = 4.58
- g = 4.9
- h = 4.75
- i = 3.85
- j = 6.36
- k = 6.39
- a = 40.59
- b = 40.59
- d = true

Boolean Value **d**: Prove(i g + f h = j k)

CAS

49	s8:v23^2-v12^2-v11^2+2*v12*v6-v6^2+2*v11*v5-v5^2=0 → $s8 : -v11^2 - v12^2 + v23^2 - v5^2 - v6^2 + 2\,v11\,v5 +$
50	s9:v24^2-v12^2-v11^2+2*v12*v8-v8^2+2*v11*v7-v7^2=0 → $s9 : -v11^2 - v12^2 + v24^2 - v7^2 - v8^2 + 2\,v11\,v7 +$
51	s10:v25^2-v10^2-v9^2+2*v10*v6-v6^2+2*v9*v5-v5^2=0 → $s10 : -v10^2 + v25^2 - v5^2 - v6^2 - v9^2 + 2\,v10\,v6 +$
52	s11:-1+v26*v25^4*v24^4-2*v26*v25^2*v24^2*v23^2*v21^2 → $s11 : v20^4\,v22^4\,v26 + v21^4\,v23^4\,v26 + v24^4\,v25^4\,v26$
53	Now we consider the following expression:
54	s1*(2*v6*v12*v21^4*v26+2*v20^2*v21^2*v22^2*v26-v21^4 → $1 = 0$
55	**Contradiction! This proves the original statement.**
56	The statement has a difficulty of degree 7.

Graphics

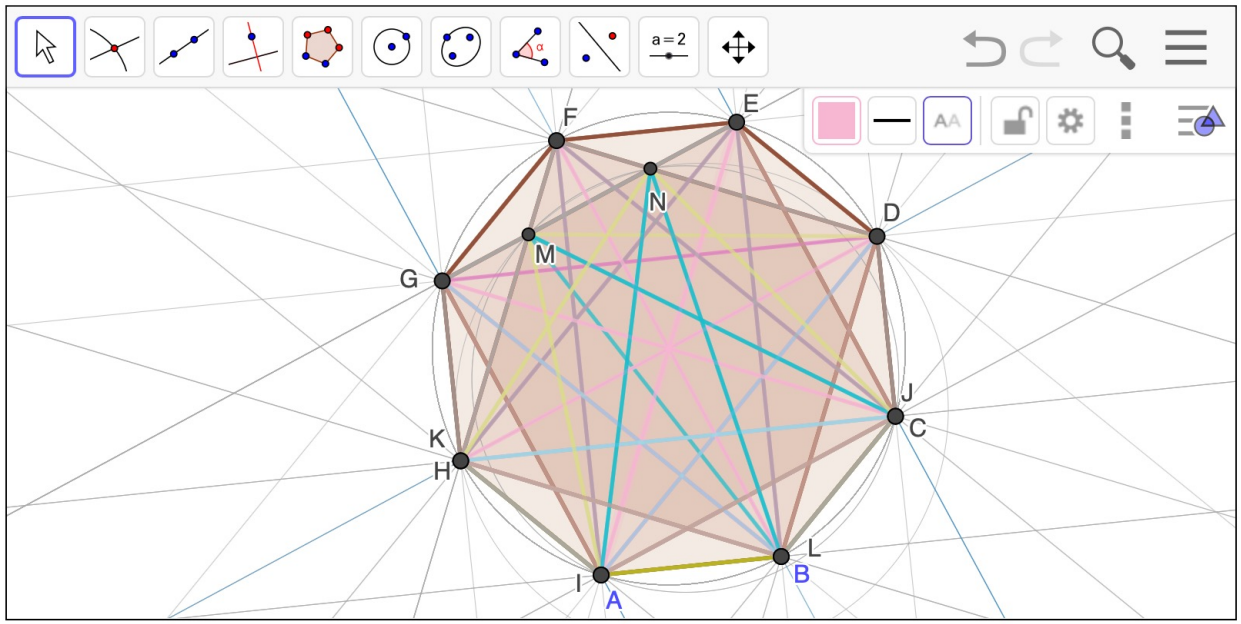


Input:

Welcome to the Automated Geometer!

Using GeoGebra 5.0.495.0 (offline).

Let us consider this initial input construction (only the visible points will be observed) :



Select relations to check:

Collinearity of three points
Equality of distances between two points
Perpendicularity of segments defined by two points
Parallelism of segments defined by two points
Concyclicity of four points

The following theorems can be proven:

1. I∈AB	294. BH=HL	590. HI=JL	872. BH⊥CJ	1153. DN⊥HM	1434. AI∥JN	1715. CJ∥GH	1965. J∈○ABI	2135. I∈○BCF	H∈○CFG
2. L∈AB	295. BH=IJ	591. HJ=JK	873. BH⊥DL	1154. DN⊥KM	1435. AI∥KL	1716. CJ∥GI	1966. K∈○ABI	2136. J∈○BCF	2314. I∈○CFG
3. I∈AC	296. BH=KL	592. HL=IJ	874. BH⊥EI	1155. EF⊥EL	1436. AI∥KM	1717. CJ∥GJ	1967. L∈○ABI	2137.	2315. J∈○CFG
4. J∈AC	297. BI=BJ	593. HL=KL	875. BH⊥FH	1156. EF⊥FI	1437. AI∥KN	1718. CJ∥GK	1968.	K∈○BCF	2316.
5. I∈AD	298. BI=CD	594. HM=KM	876. BH⊥FK	1157. EF⊥GH	1438. AI∥LM	1719. CJ∥GL	M∈○ABI	2138.	K∈○CFG

The following theorems can be proven:

1. I∈AB
2. L∈AB
3. I∈AC
4. J∈AC
5. I∈AD
6. I∈AE
7. I∈AF
8. I∈AG
9. I∈AH
10. K∈AH
11. J∈AI
12. K∈AI
13. L∈AI
14. M∈AI
15. N∈AI
16. J∈BC
17. L∈BC
18. L∈BD
19. L∈BE
20. L∈BF
21. L∈BG
22. K∈BH
23. L∈BH
24. L∈BI
25. L∈BJ
26. L∈BK
27. M∈BL
28. N∈BL
29. J∈CD
30. J∈CE
31. J∈CF
32. J∈CG
33. J∈CH
34. K∈CH
35. J∈CI
36. K∈CJ
37. L∈CJ
38. M∈CJ
39. N∈CJ
294. BH=HL
295. BH=IJ
296. BH=KL
297. BI=BJ
298. BI=CD
299. BI=CL
300. BI=DE
301. BI=DJ
302. BI=EF
303. BI=FG
304. BI=GH
305. BI=GK
306. BI=HI
307. BI=IK
308. BI=IL
309. BI=JL
310. BJ=CD
311. BJ=CL
312. BJ=DE
313. BJ=DJ
314. BJ=EF
315. BJ=FG
316. BJ=GH
317. BJ=GK
318. BJ=HI
319. BJ=IK
320. BJ=IL
321. BJ=JL
322. BK=CE
323. BK=CI
324. BK=DF
325. BK=DL
326. BK=EG
327. BK=EJ
328. BK=FH
329. BK=FK
330. BK=GI
331. BK=HL
332. BK=IJ
333. BK=KL
334. BL=CJ
590. HI=JL
591. HJ=JK
592. HL=IJ
593. HL=KL
594. HM=KM
595. HN=IM
596. HN=JN
597. HN=KN
598. IJ=KL
599. IK=IL
600. IK=JL
601. IL=JL
602. IM=JN
603. IM=KN
604. IN=JM
605. IN=LN
606. JM=LM
607. JM=LN
608. JN=KN
609. AB⊥AF
610. AB⊥AI
611. AB⊥BE
612. AB⊥BL
613. AB⊥CD
614. AB⊥CJ
615. AB⊥DJ
616. AB⊥EL
617. AB⊥FI
618. AB⊥GH
619. AB⊥GK
620. AB⊥HK
621. AC⊥AG
622. AC⊥AI
623. AC⊥BF
624. AC⊥BL
625. AC⊥CE
626. AC⊥CJ
627. AC⊥EJ
628. AC⊥FL
629. AC⊥GI
872. BH⊥CJ
873. BH⊥DL
874. BH⊥EI
875. BH⊥FH
876. BH⊥FK
877. BH⊥FM
878. BH⊥HK
879. BH⊥HM
880. BH⊥KM
881. BI⊥BL
882. BI⊥CD
883. BI⊥CJ
884. BI⊥DJ
885. BI⊥EL
886. BI⊥FI
887. BI⊥GH
888. BI⊥GK
889. BI⊥HK
890. BJ⊥BL
891. BJ⊥CF
892. BJ⊥CJ
893. BJ⊥DE
894. BJ⊥FJ
895. BJ⊥GL
896. BJ⊥HI
897. BJ⊥HK
898. BJ⊥IK
899. BK⊥BL
900. BK⊥CJ
901. BK⊥DL
902. BK⊥EI
903. BK⊥FH
904. BK⊥FK
905. BK⊥FM
906. BK⊥HK
907. BK⊥HM
908. BK⊥KM
909. BL⊥BM
910. BL⊥BN
1153. DN⊥HM
1154. DN⊥KM
1155. EF⊥EL
1156. EF⊥FI
1157. EF⊥GH
1158. EF⊥GK
1159. EF⊥HK
1160. EG⊥EJ
1161. EG⊥FL
1162. EG⊥GI
1163. EG⊥HK
1164. EH⊥FJ
1165. EH⊥GL
1166. EH⊥HI
1167. EH⊥HK
1168. EH⊥IK
1169. EI⊥FN
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1171. EI⊥HK
1172. EI⊥HL
1173. EI⊥KL
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1175. EJ⊥EN
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1180. EJ⊥MN
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1185. EK⊥IK
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1188. EL⊥IL
1189. EL⊥JK
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1191. EM⊥GI
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1442. AJ∥CJ
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1448. AJ∥GM
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1451. AJ∥IJ
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1453. AK∥BL
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1455. AK∥CJ
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1466. AL∥CK
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1721. CJ∥GN
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1723. CJ∥HJ
1724. CJ∥HK
1725. CJ∥HL
1726. CJ∥HM
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1728. CJ∥IJ
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1971. L∈○ABJ
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2140. I∈○BCG
2141. J∈○BCG
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2143. L∈○BCG
2144. I∈○BCH
2145. J∈○BCH
2146. K∈○BCH
2147. L∈○BCH
2148. J∈○BCI
2149. K∈○BCI
2150. L∈○BCI
2151. K∈○BCJ
2152. L∈○BCJ
2153. M∈○BCJ
2154. N∈○BCJ
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2156. M∈○BCL
2157. N∈○BCL
2158. F∈○BDE
2159. G∈○BDE
H∈○CFG
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2315. J∈○CFG
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2317. L∈○CFG
2318. I∈○CFH
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2320. K∈○CFH
2321. L∈○CFH
2322. J∈○CFI
2323. K∈○CFI
2324. L∈○CFI
2325. K∈○CFJ
2326. L∈○CFJ
2327. M∈○CFJ
2328. N∈○CFJ
2329. L∈○CFK
2330. I∈○CGH
2331. J∈○CGH
2332. K∈○CGH
2333. L∈○CGH
2334. J∈○CGI
2335. K∈○CGI
2336. L∈○CGI
2337. K∈○CGJ
2338. L∈○CGJ
2339. M∈○CGJ

29. J∈CD
30. J∈CE
31. J∈CF
32. J∈CG
33. J∈CH
34. K∈CH
35. J∈CI
36. K∈CJ
37. L∈CJ
38. M∈CJ
39. N∈CJ
40. N∈DF
41. K∈DH
42. M∈EG
43. N∈EG
44. K∈EH
45. N∈EM
46. K∈FH
47. M∈FH
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54. M∈HK
55. N∈HK
56. AB=AH
57. AB=AK
58. AB=AL
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63. AB=CL
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69. AB=GK

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425. CL=IL
426. CL=JL
427. CM=IN
428. CM=JM

722. AI⊥CL
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724. AI⊥CN
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727. AI⊥DG
728. AI⊥DH
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730. AI⊥DJ
731. AI⊥DK
732. AI⊥DL
733. AI⊥DM
734. AI⊥DN
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736. AI⊥EG
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739. AI⊥EJ
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754. AI⊥GJ
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757. AI⊥GM
758. AI⊥GN
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760. AI⊥HJ

1029. CI⊥HK
1030. CJ⊥CK
1031. CJ⊥CL
1032. CJ⊥CM
1033. CJ⊥CN
1034. CJ⊥DE
1035. CJ⊥DF
1036. CJ⊥DG
1037. CJ⊥DH
1038. CJ⊥DI
1039. CJ⊥DJ
1040. CJ⊥DK
1041. CJ⊥DL
1042. CJ⊥DM
1043. CJ⊥DN
1044. CJ⊥EF
1045. CJ⊥EG
1046. CJ⊥EH
1047. CJ⊥EI
1048. CJ⊥EJ
1049. CJ⊥EK
1050. CJ⊥EL
1051.

GI⊥HK
1236. GI⊥IJ
1237. GI⊥MN
1238. GJ⊥HK
1239. GJ⊥HM
1240. GJ⊥KM
1241. GK⊥HJ
1242. GK⊥HK
1243. GK⊥IL
1244. GK⊥JK
1245. GL⊥HK
1246. GL⊥JL
1247. GM⊥HK
1248. GN⊥HK
1249. HI⊥HK
1250. HI⊥JL
1251. HJ⊥HK
1252. HK⊥HL
1253. HK⊥HM
1254. HK⊥HN
1255. HK⊥IJ
1256.

BC∥HK
1490. BC∥JL
1491. BD∥BL
1492. BD∥CJ
1493. BD∥DL
1494. BD∥EI
1495. BD∥FH
1496. BD∥FK
1497. BD∥FM
1498. BD∥HK
1499. BD∥HM
1500. BD∥KM
1501. BE∥BL
1502. BE∥CD
1503. BE∥CJ
1504. BE∥DJ
1505. BE∥EL
1506. BE∥FI
1507. BE∥GH
1508. BE∥GK
1509. BE∥HK
1510. BF∥BL
1511. BF∥CE
1512. BF∥CJ
1513. BF∥EJ
1514. BF∥FL
1515. BF∥GI

1758. DE∥FJ
1759. DE∥GL
1760. DE∥HI
1761. DE∥HK
1762. DE∥IK
1763. DF∥DN
1764. DF∥FN
1765. DF∥GJ
1766. DF∥HK
1767. DF∥HL
1768. DF∥KL
1769. DG∥EF
1770. DG∥HJ
1771. DG∥HK
1772. DG∥IL
1773. DG∥JK
1774. DH∥DK
1775. DH∥EG
1776. DH∥EM
1777. DH∥EN
1778. DH∥GM
1779. DH∥GN
1780. DH∥HK
1781. DH∥IJ
1782. DH∥MN
1783. DI∥EH

1979. I∈○ACD
1980. J∈○ACD
1981. K∈○ACD
1982. L∈○ACD
1983. F∈○ACE
1984. G∈○ACE
1985. H∈○ACE
1986. I∈○ACE
1987. J∈○ACE
1988. K∈○ACE
1989. L∈○ACE
1990. G∈○ACF
1991. H∈○ACF
1992. I∈○ACF
1993. J∈○ACF
1994. K∈○ACF
1995. L∈○ACF
1996. H∈○ACG
1997. I∈○ACG
1998.

2159. G∈○BDE
2160. H∈○BDE
2161. I∈○BDE
2162. J∈○BDE
2163. K∈○BDE
2164. L∈○BDE
2165. G∈○BDF
2166. H∈○BDF
2167. I∈○BDF
2168. J∈○BDF
2169. K∈○BDF
2170. L∈○BDF
2171. H∈○BDG
2172. I∈○BDG
2173. J∈○BDG
2174. K∈○BDG
2175. L∈○BDG
2176. I∈○BDH
2177. J∈○BDH
2178.

2339. M∈○CGJ
2340. N∈○CGJ
2341. L∈○CGK
2342. J∈○CHI
2343. K∈○CHI
2344. L∈○CHI
2345. K∈○CHJ
2346. L∈○CHJ
2347. M∈○CHJ
2348. N∈○CHJ
2349. L∈○CHK
2350. M∈○CHK
2351. N∈○CHK
2352. K∈○CIJ
2353. L∈○CIJ
2354. M∈○CIJ
2355. N∈○CIJ
2356. L∈○CIK
2357. L∈○CJK
2358.

63. AB=CL
64. AB=DE
65. AB=DJ
66. AB=EF
67. AB=FG
68. AB=GH
69. AB=GK
70. AB=HI
71. AB=IK
72. AB=IL
73. AB=JL
74. AC=AG
75. AC=AJ
76. AC=BD
77. AC=BH
78. AC=BK
79. AC=CE
80. AC=CI
81. AC=DF
82. AC=DL
83. AC=EG
84. AC=EJ
85. AC=FH
86. AC=FK
87. AC=GI
88. AC=HL
89. AC=IJ
90. AC=KL
91. AD=AF
92. AD=BE
93. AD=BG
94. AD=CF
95. AD=CH
96. AD=CK
97. AD=DG
98. AD=DI
99. AD=EH
100. AD=EK
101. AD=EL
102. AD=FI
103. AD=FJ
104. AD=GL
105. AD=HJ
106. AD=JK
107. AE=BF
108. AE=CG
109. AE=DH
110. AE=DK
111. AE=EI
112. AE=FL

341. BM=LN
342. BN=CM
343. BN=IN
344. BN=JM
345. BN=LN
346. CD=CL
347. CD=DE
348. CD=DJ
349. CD=EF
350. CD=FG
351. CD=GH
352. CD=GK
353. CD=HI
354. CD=IK
355. CD=IL
356. CD=JL
357. CE=CI
358. CE=DF
359. CE=DL
360. CE=EG
361. CE=EJ
362. CE=FH
363. CE=FK
364. CE=GI
365. CE=HL
366. CE=IJ
367. CE=KL
368. CF=CH
369. CF=CK
370. CF=DG
371. CF=DI
372. CF=EH
373. CF=EK
374. CF=EL
375. CF=FI
376. CF=FJ
377. CF=GL
378. CF=HJ
379. CF=JK
380. CG=DH
381. CG=DK
382. CG=EI
383. CG=FL
384. CG=GJ
385. CH=CK
386. CH=DG
387. CH=DI
388. CH=EH
389. CH=EK
390. CH=EL

618. AB⊥GH
619. AB⊥GK
620. AB⊥HK
621. AC⊥AG
622. AC⊥AI
623. AC⊥BF
624. AC⊥BL
625. AC⊥CE
626. AC⊥CJ
627. AC⊥EJ
628. AC⊥FL
629. AC⊥GI
630. AC⊥HK
631. AD⊥AH
632. AD⊥AI
633. AD⊥AK
634. AD⊥BG
635. AD⊥BL
636. AD⊥CF
637. AD⊥CJ
638. AD⊥DE
639. AD⊥FJ
640. AD⊥GL
641. AD⊥HI
642. AD⊥HK
643. AD⊥IK
644. AE⊥AI
645. AE⊥BH
646. AE⊥BK
647. AE⊥BL
648. AE⊥CG
649. AE⊥CJ
650. AE⊥DF
651. AE⊥DN
652. AE⊥FN
653. AE⊥GJ
654. AE⊥HK
655. AE⊥HL
656. AE⊥KL
657. AF⊥AI
658. AF⊥AL
659. AF⊥BI
660. AF⊥BL
661. AF⊥CH
662. AF⊥CJ
663. AF⊥CK
664. AF⊥DG
665. AF⊥EF

881. BI⊥BL
882. BI⊥CD
883. BI⊥CJ
884. BI⊥DJ
885. BI⊥EL
886. BI⊥FI
887. BI⊥GH
888. BI⊥GK
889. BI⊥HK
890. BJ⊥BL
891. BJ⊥CF
892. BJ⊥CJ
893. BJ⊥DE
894. BJ⊥FJ
895. BJ⊥GL
896. BJ⊥HI
897. BJ⊥HK
898. BJ⊥IK
899. BK⊥BL
900. BK⊥CJ
901. BK⊥DL
902. BK⊥EI
903. BK⊥FH
904. BK⊥FK
905. BK⊥FM
906. BK⊥HK
907. BK⊥HM
908. BK⊥KM
909. BL⊥BM
910. BL⊥BN
911. BL⊥CD
912. BL⊥CE
913. BL⊥CF
914. BL⊥CG
915. BL⊥CH
916. BL⊥CI
917. BL⊥CJ
918. BL⊥CK
919. BL⊥CL
920. BL⊥CM
921. BL⊥CN
922. BL⊥DE
923. BL⊥DF
924. BL⊥DG
925. BL⊥DH
926. BL⊥DI
927. BL⊥DJ
928. BL⊥DK

1145. DL⊥HL
1146. DL⊥KL
1147. DM⊥HK
1148. DN⊥EI
1149. DN⊥FH
1150. DN⊥FK
1151. DN⊥FM
1152. DN⊥HK
1153. DN⊥HM
1154. DN⊥KM
1155. EF⊥EL
1156. EF⊥FI
1157. EF⊥GH
1158. EF⊥GK
1159. EF⊥HK
1160. EG⊥EJ
1161. EG⊥FL
1162. EG⊥GI
1163. EG⊥HK
1164. EH⊥FJ
1165. EH⊥GL
1166. EH⊥HI
1167. EH⊥HK
1168. EH⊥IK
1169. EI⊥FN
1170. EI⊥GJ
1171. EI⊥HK
1172. EI⊥HL
1173. EI⊥KL
1174. EJ⊥EM
1175. EJ⊥EN
1176. EJ⊥GM
1177. EJ⊥GN
1178. EJ⊥HK
1179. EJ⊥IJ
1180. EJ⊥MN
1181. EK⊥FJ
1182. EK⊥GL
1183. EK⊥HI
1184. EK⊥HK
1185. EK⊥IK
1186. EL⊥HJ
1187. EL⊥HK
1188. EL⊥IL
1189. EL⊥JK
1190. EM⊥FL
1191. EM⊥GI
1192. EM⊥HK

1409. AI∥FK
1410. AI∥FL
1411. AI∥FM
1412. AI∥FN
1413. AI∥GH
1414. AI∥GI
1415. AI∥GJ
1416. AI∥GK
1417. AI∥GL
1418. AI∥GM
1419. AI∥GN
1420. AI∥HI
1421. AI∥HJ
1422. AI∥HK
1423. AI∥HL
1424. AI∥HM
1425. AI∥HN
1426. AI∥IJ
1427. AI∥IK
1428. AI∥IL
1429. AI∥IM
1430. AI∥IN
1431. AI∥JK
1432. AI∥JL
1433. AI∥JM
1434. AI∥JN
1435. AI∥KL
1436. AI∥KM
1437. AI∥KN
1438. AI∥LM
1439. AI∥LN
1440. AJ∥BL
1441. AJ∥CI
1442. AJ∥CJ
1443. AJ∥DH
1444. AJ∥DK
1445. AJ∥EG
1446. AJ∥EM
1447. AJ∥EN
1448. AJ∥GM
1449. AJ∥GN
1450. AJ∥HK
1451. AJ∥IJ
1452. AK∥BG
1453. AK∥BL
1454. AK∥CF
1455. AK∥CJ
1456. AK∥DE

1673. CH∥JK
1674. CI∥CJ
1675. CI∥DH
1676. CI∥DK
1677. CI∥EG
1678. CI∥EM
1679. CI∥EN
1680. CI∥GM
1681. CI∥GN
1682. CI∥HK
1683. CI∥IJ
1684. CJ∥CK
1685. CJ∥CL
1686. CJ∥CM
1687. CJ∥CN
1688. CJ∥DE
1689. CJ∥DF
1690. CJ∥DG
1691. CJ∥DH
1692. CJ∥DI
1693. CJ∥DJ
1694. CJ∥DK
1695. CJ∥DL
1696. CJ∥DM
1697. CJ∥DN
1698. CJ∥EF
1699. CJ∥EG
1700. CJ∥EH
1701. CJ∥EI
1702. CJ∥EJ
1703. CJ∥EK
1704. CJ∥EL
1705. CJ∥EM
1706. CJ∥EN
1707. CJ∥FG
1708. CJ∥FH
1709. CJ∥FI
1710. CJ∥FJ
1711. CJ∥FK
1712. CJ∥FL
1713. CJ∥FM
1714. CJ∥FN
1715. CJ∥GH
1716. CJ∥GI
1717. CJ∥GJ
1718. CJ∥GK
1719. CJ∥GL
1720. CJ∥GM

K∈○ABC
1934. L∈○ABC
1935. E∈○ABD
1936. F∈○ABD
1937. G∈○ABD
1938. H∈○ABD
1939. I∈○ABD
1940. J∈○ABD
1941. K∈○ABD
1942. L∈○ABD
1943. F∈○ABE
1944. G∈○ABE
1945. H∈○ABE
1946. I∈○ABE
1947. J∈○ABE
1948. K∈○ABE
1949. L∈○ABE
1950. G∈○ABF
1951. H∈○ABF
1952. I∈○ABF
1953. J∈○ABF
1954. K∈○ABF
1955. L∈○ABF
1956. H∈○ABG
1957. I∈○ABG
1958. J∈○ABG
1959. K∈○ABG
1960. L∈○ABG
1961. I∈○ABH
1962. J∈○ABH
1963. K∈○ABH
1964. L∈○ABH
1965. J∈○ABI
1966. K∈○ABI

2131. K∈○BCE
2132. L∈○BCE
2133. G∈○BCF
2134. H∈○BCF
2135. I∈○BCF
2136. J∈○BCF
2137. K∈○BCF
2138. L∈○BCF
2139. H∈○BCG
2140. I∈○BCG
2141. J∈○BCG
2142. K∈○BCG
2143. L∈○BCG
2144. I∈○BCH
2145. J∈○BCH
2146. K∈○BCH
2147. L∈○BCH
2148. J∈○BCI
2149. K∈○BCI
2150. L∈○BCI
2151. K∈○BCJ
2152. L∈○BCJ
2153. M∈○BCJ
2154. N∈○BCJ
2155. L∈○BCK
2156. M∈○BCL
2157. N∈○BCL
2158. F∈○BDE
2159. G∈○BDE
2160. H∈○BDE
2161. I∈○BDE
2162. J∈○BDE
2163. K∈○BDE
2164. L∈○BDE
2165. G∈○BDF
2166. H∈○BDF
2167. I∈○BDF
2168. J∈○BDF
2169. K∈○BDF

2340. N∈○CGJ
2341. L∈○CGK
2342. J∈○CHI
2343. K∈○CHI
2344. L∈○CHI
2345. K∈○CHJ
2346. L∈○CHJ
2347. M∈○CHJ
2348. N∈○CHJ
2349. L∈○CHK
2350. M∈○CHK
2351. N∈○CHK
2352. K∈○CIJ
2353. L∈○CIJ
2354. M∈○CIJ
2355. N∈○CIJ
2356. L∈○CIK
2357. L∈○CJK
2358. M∈○CJK
2359. N∈○CJK
2360. M∈○CJL
2361. N∈○CJL
2362. G∈○DEF
2363. H∈○DEF
2364. I∈○DEF
2365. J∈○DEF
2366. K∈○DEF
2367. L∈○DEF
2368. H∈○DEG
2369. I∈○DEG
2370. J∈○DEG
2371. K∈○DEG
2372. L∈○DEG
2373. I∈○DEH
2374. J∈○DEH
2375. K∈○DEH
2376. L∈○DEH
2377. J∈○DEI
2378. K∈○DEI

112. AE=FL
113. AE=GJ
114. AF=BE
115. AF=BG
116. AF=CF
117. AF=CH
118. AF=CK
119. AF=DG
120. AF=DI
121. AF=EH
122. AF=EK
123. AF=EL
124. AF=FI
125. AF=FJ
126. AF=GL
127. AF=HJ
128. AF=JK
129. AG=AJ
130. AG=BD
131. AG=BH
132. AG=BK
133. AG=CE
134. AG=CI
135. AG=DF
136. AG=DL
137. AG=EG
138. AG=EJ
139. AG=FH
140. AG=FK
141. AG=GI
142. AG=HL
143. AG=IJ
144. AG=KL
145. AH=AK
146. AH=AL
147. AH=BC
148. AH=BI
149. AH=BJ
150. AH=CD
151. AH=CL
152. AH=DE
153. AH=DJ
154. AH=EF
155. AH=FG
156. AH=GH
157. AH=GK
158. AH=HI
159. AH=IK
160. AH=IL
161. AH=IJ

390. CH=EL
391. CH=FI
392. CH=FJ
393. CH=GL
394. CH=HJ
395. CH=JK
396. CI=DF
397. CI=DL
398. CI=EG
399. CI=EJ
400. CI=FH
401. CI=FK
402. CI=GI
403. CI=HL
404. CI=IJ
405. CI=KL
406. CJ=HK
407. CK=DG
408. CK=DI
409. CK=EH
410. CK=EK
411. CK=EL
412. CK=FI
413. CK=FJ
414. CK=GL
415. CK=HJ
416. CK=JK
417. CL=DE
418. CL=DJ
419. CL=EF
420. CL=FG
421. CL=GH
422. CL=GK
423. CL=HI
424. CL=IK
425. CL=IL
426. CL=JL
427. CM=IN
428. CM=JM
429. CM=LM
430. CM=LN
431. CN=DM
432. CN=HN
433. CN=IM
434. CN=JN
435. CN=KN
436. DE=DJ
437. DE=EF
438. DE=FG
439. DE=GH

665. AF⊥EF
666. AF⊥HJ
667. AF⊥HK
668. AF⊥IL
669. AF⊥JK
670. AG⊥AI
671. AG⊥AJ
672. AG⊥BL
673. AG⊥CI
674. AG⊥CJ
675. AG⊥DH
676. AG⊥DK
677. AG⊥EG
678. AG⊥EM
679. AG⊥EN
680. AG⊥GM
681. AG⊥GN
682. AG⊥HK
683. AG⊥IJ
684. AG⊥MN
685. AH⊥AI
686. AH⊥BC
687. AH⊥BJ
688. AH⊥BL
689. AH⊥CJ
690. AH⊥CL
691. AH⊥DI
692. AH⊥EH
693. AH⊥EK
694. AH⊥FG
695. AH⊥HK
696. AH⊥JL
697. AI⊥AJ
698. AI⊥AK
699. AI⊥AL
700. AI⊥AM
701. AI⊥AN
702. AI⊥BC
703. AI⊥BD
704. AI⊥BE
705. AI⊥BF
706. AI⊥BG
707. AI⊥BH
708. AI⊥BI
709. AI⊥BJ
710. AI⊥BK
711. AI⊥BL

928. BL⊥DK
929. BL⊥DL
930. BL⊥DM
931. BL⊥DN
932. BL⊥EF
933. BL⊥EG
934. BL⊥EH
935. BL⊥EI
936. BL⊥EJ
937. BL⊥EK
938. BL⊥EL
939. BL⊥EM
940. BL⊥EN
941. BL⊥FG
942. BL⊥FH
943. BL⊥FI
944. BL⊥FJ
945. BL⊥FK
946. BL⊥FL
947. BL⊥FM
948. BL⊥FN
949. BL⊥GH
950. BL⊥GI
951. BL⊥GJ
952. BL⊥GK
953. BL⊥GL
954. BL⊥GM
955. BL⊥GN
956. BL⊥HI
957. BL⊥HJ
958. BL⊥HK
959. BL⊥HL
960. BL⊥HM
961. BL⊥HN
962. BL⊥IJ
963. BL⊥IK
964. BL⊥IL
965. BL⊥IM
966. BL⊥IN
967. BL⊥JK
968. BL⊥JL
969. BL⊥JM
970. BL⊥JN
971. BL⊥KL
972. BL⊥KM
973. BL⊥KN
974. BL⊥LM

1192. EM⊥HK
1193. EN⊥FL
1194. EN⊥GI
1195. EN⊥HK
1196. FG⊥FJ
1197. FG⊥GL
1198. FG⊥HI
1199. FG⊥HK
1200. FG⊥IK
1201. FH⊥FN
1202. FH⊥GJ
1203. FH⊥HK
1204. FH⊥HL
1205. FH⊥KL
1206. FI⊥HJ
1207. FI⊥HK
1208. FI⊥IL
1209. FI⊥JK
1210. FJ⊥HK
1211. FJ⊥JL
1212. FK⊥FN
1213. FK⊥GJ
1214. FK⊥HK
1215. FK⊥HL
1216. FK⊥KL
1217. FL⊥GM
1218. FL⊥GN
1219. FL⊥HK
1220. FL⊥IJ
1221. FM⊥FN
1222. FM⊥GJ
1223. FM⊥HK
1224. FM⊥HL
1225. FM⊥KL
1226. FN⊥HK
1227. FN⊥HM
1228. FN⊥KM
1229. GH⊥HJ
1230. GH⊥HK
1231. GH⊥IL
1232. GH⊥JK
1233. GI⊥GM
1234. GI⊥GN
1235. GI⊥HK
1236. GI⊥IJ
1237. GI⊥MN
1238. GJ⊥HK

1456. AK∥DE
1457. AK∥FJ
1458. AK∥GL
1459. AK∥HI
1460. AK∥HK
1461. AK∥IK
1462. AL∥BI
1463. AL∥BL
1464. AL∥CH
1465. AL∥CJ
1466. AL∥CK
1467. AL∥DG
1468. AL∥EF
1469. AL∥HJ
1470. AL∥HK
1471. AL∥IL
1472. AL∥JK
1473. AM∥BL
1474. AM∥CJ
1475. AM∥HK
1476. AM∥IM
1477. AN∥BL
1478. AN∥CJ
1479. AN∥HK
1480. AN∥IN
1481. BC∥BJ
1482. BC∥BL
1483. BC∥CJ
1484. BC∥CL
1485. BC∥DI
1486. BC∥EH
1487. BC∥EK
1488. BC∥FG
1489. BC∥HK
1490. BC∥JL
1491. BD∥BL
1492. BD∥CJ
1493. BD∥DL
1494. BD∥EI
1495. BD∥FH
1496. BD∥FK
1497. BD∥FM
1498. BD∥HK
1499. BD∥HM
1500. BD∥KM
1501. BE∥BL
1502. BE∥CD

1720. CJ∥GM
1721. CJ∥GN
1722. CJ∥HI
1723. CJ∥HJ
1724. CJ∥HK
1725. CJ∥HL
1726. CJ∥HM
1727. CJ∥HN
1728. CJ∥IJ
1729. CJ∥IK
1730. CJ∥IL
1731. CJ∥IM
1732. CJ∥IN
1733. CJ∥JK
1734. CJ∥JL
1735. CJ∥JM
1736. CJ∥JN
1737. CJ∥KL
1738. CJ∥KM
1739. CJ∥KN
1740. CJ∥LM
1741. CJ∥LN
1742. CK∥DG
1743. CK∥EF
1744. CK∥HJ
1745. CK∥HK
1746. CK∥IL
1747. CK∥JK
1748. CL∥DI
1749. CL∥EH
1750. CL∥EK
1751. CL∥FG
1752. CL∥HK
1753. CL∥JL
1754. CM∥HK
1755. CM∥JM
1756. CN∥HK
1757. CN∥JN
1758. DE∥FJ
1759. DE∥GL
1760. DE∥HI
1761. DE∥HK
1762. DE∥IK
1763. DF∥DN
1764. DF∥FN
1765. DF∥GJ
1766. DF∥HK

1966. K∈○ABI
1967. L∈○ABI
1968. M∈○ABI
1969. N∈○ABI
1970. K∈○ABJ
1971. L∈○ABJ
1972. L∈○ABK
1973. M∈○ABL
1974. N∈○ABL
1975. E∈○ACD
1976. F∈○ACD
1977. G∈○ACD
1978. H∈○ACD
1979. I∈○ACD
1980. J∈○ACD
1981. K∈○ACD
1982. L∈○ACD
1983. F∈○ACE
1984. G∈○ACE
1985. H∈○ACE
1986. I∈○ACE
1987. J∈○ACE
1988. K∈○ACE
1989. L∈○ACE
1990. G∈○ACF
1991. H∈○ACF
1992. I∈○ACF
1993. J∈○ACF
1994. K∈○ACF
1995. L∈○ACF
1996. H∈○ACG
1997. I∈○ACG
1998. J∈○ACG
1999.

2169. K∈○BDF
2170. L∈○BDF
2171. H∈○BDG
2172. I∈○BDG
2173. J∈○BDG
2174. K∈○BDG
2175. L∈○BDG
2176. I∈○BDH
2177. J∈○BDH
2178. K∈○BDH
2179. L∈○BDH
2180. J∈○BDI
2181. K∈○BDI
2182. L∈○BDI
2183. K∈○BDJ
2184. L∈○BDJ
2185. L∈○BDK
2186. M∈○BDL
2187. N∈○BDL
2188. G∈○BEF
2189. H∈○BEF
2190. I∈○BEF
2191. J∈○BEF
2192. K∈○BEF
2193. L∈○BEF
2194. H∈○BEG
2195. I∈○BEG
2196. J∈○BEG
2197. K∈○BEG
2198. L∈○BEG
2199. I∈○BEH
2200. J∈○BEH
2201. K∈○BEH
2202. L∈○BEH
2203. J∈○BEI

2378. K∈○DEI
2379. L∈○DEI
2380. K∈○DEJ
2381. L∈○DEJ
2382. L∈○DEK
2383. H∈○DFG
2384. I∈○DFG
2385. J∈○DFG
2386. K∈○DFG
2387. L∈○DFG
2388. I∈○DFH
2389. J∈○DFH
2390. K∈○DFH
2391. L∈○DFH
2392. J∈○DFI
2393. K∈○DFI
2394. L∈○DFI
2395. K∈○DFJ
2396. L∈○DFJ
2397. L∈○DFK
2398. I∈○DGH
2399. J∈○DGH
2400. K∈○DGH
2401. L∈○DGH
2402. J∈○DGI
2403. K∈○DGI
2404. L∈○DGI
2405. K∈○DGJ
2406. L∈○DGJ
2407. L∈○DGK
2408. J∈○DHI
2409. K∈○DHI
2410. L∈○DHI
2411. K∈○DHJ
2412. L∈○DHJ
2413. L∈○DHK
2414. M∈○DHK

161. AH=JL
162. AI=BL
163. AI=CJ
164. AI=HK
165. AJ=BD
166. AJ=BH
167. AJ=BK
168. AJ=CE
169. AJ=CI
170. AJ=DF
171. AJ=DL
172. AJ=EG
173. AJ=EJ
174. AJ=FH
175. AJ=FK
176. AJ=GI
177. AJ=HL
178. AJ=IJ
179. AJ=KL
180. AK=AL
181. AK=BC
182. AK=BI
183. AK=BJ
184. AK=CD
185. AK=CL
186. AK=DE
187. AK=DJ
188. AK=EF
189. AK=FG
190. AK=GH
191. AK=GK
192. AK=HI
193. AK=IK
194. AK=IL
195. AK=JL
196. AL=BC
197. AL=BI
198. AL=BJ
199. AL=CD
200. AL=CL
201. AL=DE
202. AL=DJ
203. AL=EF
204. AL=FG
205. AL=GH
206. AL=GK
207. AL=HI
208. AL=IK
209. AL=IL
210. AL=JL
439. DE=GH
440. DE=GK
441. DE=HI
442. DE=IK
443. DE=IL
444. DE=JL
445. DF=DL
446. DF=EG
447. DF=EJ
448. DF=FH
449. DF=FK
450. DF=GI
451. DF=HL
452. DF=IJ
453. DF=KL
454. DG=DI
455. DG=EH
456. DG=EK
457. DG=EL
458. DG=FI
459. DG=FJ
460. DG=GL
461. DG=HJ
462. DG=JK
463. DH=DK
464. DH=EI
465. DH=FL
466. DH=GJ
467. DI=EH
468. DI=EK
469. DI=EL
470. DI=FI
471. DI=FJ
472. DI=GL
473. DI=HJ
474. DI=JK
475. DJ=EF
476. DJ=FG
477. DJ=GH
478. DJ=GK
479. DJ=HI
480. DJ=IK
481. DJ=IL
482. DJ=JL
483. DK=EI
484. DK=FL
485. DK=GJ
486. DL=EG
487. DL=EJ
488. DL=FH
712. AI⊥BM
713. AI⊥BN
714. AI⊥CD
715. AI⊥CE
716. AI⊥CF
717. AI⊥CG
718. AI⊥CH
719. AI⊥CI
720. AI⊥CJ
721. AI⊥CK
722. AI⊥CL
723. AI⊥CM
724. AI⊥CN
725. AI⊥DE
726. AI⊥DF
727. AI⊥DG
728. AI⊥DH
729. AI⊥DI
730. AI⊥DJ
731. AI⊥DK
732. AI⊥DL
733. AI⊥DM
734. AI⊥DN
735. AI⊥EF
736. AI⊥EG
737. AI⊥EH
738. AI⊥EI
739. AI⊥EJ
740. AI⊥EK
741. AI⊥EL
742. AI⊥EM
743. AI⊥EN
744. AI⊥FG
745. AI⊥FH
746. AI⊥FI
747. AI⊥FJ
748. AI⊥FK
749. AI⊥FL
750. AI⊥FM
751. AI⊥FN
752. AI⊥GH
753. AI⊥GI
754. AI⊥GJ
755. AI⊥GK
756. AI⊥GL
757. AI⊥GM
758. AI⊥GN
975. BL⊥LN
976. BM⊥CJ
977. BM⊥HK
978. BN⊥CJ
979. BN⊥HK
980. CD⊥CH
981. CD⊥CJ
982. CD⊥CK
983. CD⊥DG
984. CD⊥EF
985. CD⊥HJ
986. CD⊥HK
987. CD⊥IL
988. CD⊥JK
989. CE⊥CI
990. CE⊥CJ
991. CE⊥DH
992. CE⊥DK
993. CE⊥EG
994. CE⊥EM
995. CE⊥EN
996. CE⊥GM
997. CE⊥GN
998. CE⊥HK
999. CE⊥IJ
1000. CE⊥MN
1001. CF⊥CJ
1002. CF⊥CL
1003. CF⊥DI
1004. CF⊥EH
1005. CF⊥EK
1006. CF⊥FG
1007. CF⊥HK
1008. CF⊥JL
1009. CG⊥CJ
1010. CG⊥DL
1011. CG⊥EI
1012. CG⊥FH
1013. CG⊥FK
1014. CG⊥FM
1015. CG⊥HK
1016. CG⊥HM
1017. CG⊥KM
1018. CH⊥CJ
1019. CH⊥DJ
1020. CH⊥EL
1021. CH⊥FI
1239. GJ⊥HM
1240. GJ⊥KM
1241. GK⊥HJ
1242. GK⊥HK
1243. GK⊥IL
1244. GK⊥JK
1245. GL⊥HK
1246. GL⊥JL
1247. GM⊥HK
1248. GN⊥HK
1249. HI⊥HK
1250. HI⊥JL
1251. HJ⊥HK
1252. HK⊥HL
1253. HK⊥HM
1254. HK⊥HN
1255. HK⊥IJ
1256. HK⊥IK
1257. HK⊥IL
1258. HK⊥IM
1259. HK⊥IN
1260. HK⊥JK
1261. HK⊥JL
1262. HK⊥JM
1263. HK⊥JN
1264. HK⊥KL
1265. HK⊥KM
1266. HK⊥KN
1267. HK⊥LM
1268. HK⊥LN
1269. HL⊥HM
1270. HL⊥KM
1271. HM⊥KL
1272. IK⊥JL
1273. KL⊥KM
1274. AB∦AI
1275. AB∦AL
1276. AB∦BI
1277. AB∦BL
1278. AB∦CH
1279. AB∦CJ
1280. AB∦CK
1281. AB∦DG
1282. AB∦EF
1283. AB∦HJ
1284. AB∦HK
1285. AB∦IL
1503. BE∦CJ
1504. BE∦DJ
1505. BE∦EL
1506. BE∦FI
1507. BE∦GH
1508. BE∦GK
1509. BE∦HK
1510. BF∦BL
1511. BF∦CE
1512. BF∦CJ
1513. BF∦EJ
1514. BF∦FL
1515. BF∦GI
1516. BF∦HK
1517. BG∦BL
1518. BG∦CF
1519. BG∦CJ
1520. BG∦DE
1521. BG∦FJ
1522. BG∦GL
1523. BG∦HI
1524. BG∦HK
1525. BG∦IK
1526. BH∦BK
1527. BH∦BL
1528. BH∦CG
1529. BH∦CJ
1530. BH∦DF
1531. BH∦DN
1532. BH∦FN
1533. BH∦GJ
1534. BH∦HK
1535. BH∦HL
1536. BH∦KL
1537. BI∦BL
1538. BI∦CH
1539. BI∦CJ
1540. BI∦CK
1541. BI∦DG
1542. BI∦EF
1543. BI∦HJ
1544. BI∦HK
1545. BI∦IL
1546. BI∦JK
1547. BJ∦BL
1548. BJ∦CJ
1549. BJ∦CL
1767. DF∦HL
1768. DF∦KL
1769. DG∦EF
1770. DG∦HJ
1771. DG∦HK
1772. DG∦IL
1773. DG∦JK
1774. DH∦DK
1775. DH∦EG
1776. DH∦EM
1777. DH∦EN
1778. DH∦GM
1779. DH∦GN
1780. DH∦HK
1781. DH∦IJ
1782. DH∦MN
1783. DI∦EH
1784. DI∦EK
1785. DI∦FG
1786. DI∦HK
1787. DI∦JL
1788. DJ∦EL
1789. DJ∦FI
1790. DJ∦GH
1791. DJ∦GK
1792. DJ∦HK
1793. DK∦EG
1794. DK∦EM
1795. DK∦EN
1796. DK∦GM
1797. DK∦GN
1798. DK∦HK
1799. DK∦IJ
1800. DL∦EI
1801. DL∦FH
1802. DL∦FK
1803. DL∦FM
1804. DL∦HK
1805. DL∦HM
1806. DL∦KM
1807. DM∦HK
1808. DN∦FN
1809. DN∦GJ
1810. DN∦HK
1811. DN∦HL
1812. DN∦KL
1813. EF∦HJ
1999. K∈○ACG
2000. L∈○ACG
2001. I∈○ACH
2002. J∈○ACH
2003. K∈○ACH
2004. L∈○ACH
2005. J∈○ACI
2006. K∈○ACI
2007. L∈○ACI
2008. M∈○ACI
2009. N∈○ACI
2010. K∈○ACJ
2011. L∈○ACJ
2012. M∈○ACJ
2013. N∈○ACJ
2014. L∈○ACK
2015. N∈○ACM
2016. F∈○ADE
2017. G∈○ADE
2018. H∈○ADE
2019. I∈○ADE
2020. J∈○ADE
2021. K∈○ADE
2022. L∈○ADE
2023. G∈○ADF
2024. H∈○ADF
2025. I∈○ADF
2026. J∈○ADF
2027. K∈○ADF
2028. L∈○ADF
2029. H∈○ADG
2030. I∈○ADG
2031. J∈○ADG
2032.
2204. K∈○BEI
2205. L∈○BEI
2206. K∈○BEJ
2207. L∈○BEJ
2208. L∈○BEK
2209. M∈○BEL
2210. N∈○BEL
2211. H∈○BFG
2212. I∈○BFG
2213. J∈○BFG
2214. K∈○BFG
2215. L∈○BFG
2216. I∈○BFH
2217. J∈○BFH
2218. K∈○BFH
2219. L∈○BFH
2220. J∈○BFI
2221. K∈○BFI
2222. L∈○BFI
2223. K∈○BFJ
2224. L∈○BFJ
2225. L∈○BFK
2226. M∈○BFL
2227. N∈○BFL
2228. I∈○BGH
2229. J∈○BGH
2230. K∈○BGH
2231. L∈○BGH
2232. J∈○BGI
2233. K∈○BGI
2234. L∈○BGI
2235. K∈○BGJ
2236. L∈○BGJ
2237. L∈○BGK
2238. M∈○BGL
2239. N∈○BGL
2240. J∈○BHI
2241. K∈○BHI
2242. L∈○BHI
2415. N∈○DHK
2416. N∈○DHM
2417. K∈○DIJ
2418. L∈○DIJ
2419. L∈○DIK
2420. L∈○DJK
2421. H∈○EFG
2422. I∈○EFG
2423. J∈○EFG
2424. K∈○EFG
2425. L∈○EFG
2426. I∈○EFH
2427. J∈○EFH
2428. K∈○EFH
2429. L∈○EFH
2430. J∈○EFI
2431. K∈○EFI
2432. L∈○EFI
2433. K∈○EFJ
2434. L∈○EFJ
2435. L∈○EFK
2436. I∈○EGH
2437. J∈○EGH
2438. K∈○EGH
2439. L∈○EGH
2440. J∈○EGI
2441. K∈○EGI
2442. L∈○EGI
2443. K∈○EGJ
2444. L∈○EGJ
2445. L∈○EGK
2446. N∈○EGM
2447. J∈○EHI
2448. K∈○EHI
2449. L∈○EHI
2450. K∈○EHJ
2451. L∈○EHJ
2452. L∈○EHK
2453. M∈○EHK
2454. N∈○EHK

209. AL=IL
210. AL=JL
211. AM=CN
212. AM=DM
213. AM=HN
214. AM=IM
215. AM=JN
216. AM=KN
217. AN=BM
218. AN=BN
219. AN=CM
220. AN=IN
221. AN=JM
222. AN=LN
223. BC=BI
224. BC=BJ
225. BC=CD
226. BC=CL
227. BC=DE
228. BC=DJ
229. BC=EF
230. BC=FG
231. BC=GH
232. BC=GK
233. BC=HI
234. BC=IK
235. BC=IL
236. BC=JL
237. BD=BH
238. BD=BK
239. BD=CE
240. BD=CI
241. BD=DF
242. BD=DL
243. BD=EG
244. BD=EJ
245. BD=FH
246. BD=FK
247. BD=GI
248. BD=HL
249. BD=IJ
250. BD=KL
251. BE=BG
252. BE=CF
253. BE=CH
254. BE=CK
255. BE=DG
256. BE=DI
257. BE=EH
258. BE=EK
259. BE=EI

487. DL=EJ
488. DL=FH
489. DL=FK
490. DL=GI
491. DL=HL
492. DL=IJ
493. DL=KL
494. DM=HN
495. DM=IM
496. DM=JN
497. DM=KN
498. DN=EM
499. DN=GN
500. DN=HM
501. EF=FG
502. EF=GH
503. EF=GK
504. EF=HI
505. EF=IK
506. EF=IL
507. EF=JL
508. EG=EJ
509. EG=FH
510. EG=FK
511. EG=GI
512. EG=HL
513. EG=IJ
514. EG=KL
515. EH=EK
516. EH=EL
517. EH=FI
518. EH=FJ
519. EH=GL
520. EH=HJ
521. EH=JK
522. EI=FL
523. EI=GJ
524. EJ=FH
525. EJ=FK
526. EJ=GI
527. EJ=HL
528. EJ=IJ
529. EJ=KL
530. EK=EL
531. EK=FI
532. EK=FJ
533. EK=GL
534. EK=HJ
535. EK=JK
536. EL=FI
537. EL=FJ

757. AI⊥GM
758. AI⊥GN
759. AI⊥HI
760. AI⊥HJ
761. AI⊥HK
762. AI⊥HL
763. AI⊥HM
764. AI⊥HN
765. AI⊥IJ
766. AI⊥IK
767. AI⊥IL
768. AI⊥IM
769. AI⊥IN
770. AI⊥JK
771. AI⊥JL
772. AI⊥JM
773. AI⊥JN
774. AI⊥KL
775. AI⊥KM
776. AI⊥KN
777. AI⊥LM
778. AI⊥LN
779. AI⊥MN
780. AJ⊥BF
781. AJ⊥BL
782. AJ⊥CE
783. AJ⊥CJ
784. AJ⊥EJ
785. AJ⊥FL
786. AJ⊥GI
787. AJ⊥HK
788. AK⊥BC
789. AK⊥BJ
790. AK⊥BL
791. AK⊥CJ
792. AK⊥CL
793. AK⊥DI
794. AK⊥EH
795. AK⊥EK
796. AK⊥FG
797. AK⊥HK
798. AK⊥JL
799. AL⊥BE
800. AL⊥BL
801. AL⊥CD
802. AL⊥CJ
803. AL⊥DJ
804. AL⊥EL

1020. CH⊥EL
1021. CH⊥FI
1022. CH⊥GH
1023. CH⊥GK
1024. CH⊥HK
1025. CI⊥CJ
1026. CI⊥EJ
1027. CI⊥FL
1028. CI⊥GI
1029. CI⊥HK
1030. CJ⊥CK
1031. CJ⊥CL
1032. CJ⊥CM
1033. CJ⊥CN
1034. CJ⊥DE
1035. CJ⊥DF
1036. CJ⊥DG
1037. CJ⊥DH
1038. CJ⊥DI
1039. CJ⊥DJ
1040. CJ⊥DK
1041. CJ⊥DL
1042. CJ⊥DM
1043. CJ⊥DN
1044. CJ⊥EF
1045. CJ⊥EG
1046. CJ⊥EH
1047. CJ⊥EI
1048. CJ⊥EJ
1049. CJ⊥EK
1050. CJ⊥EL
1051. CJ⊥EM
1052. CJ⊥EN
1053. CJ⊥FG
1054. CJ⊥FH
1055. CJ⊥FI
1056. CJ⊥FJ
1057. CJ⊥FK
1058. CJ⊥FL
1059. CJ⊥FM
1060. CJ⊥FN
1061. CJ⊥GH
1062. CJ⊥GI
1063. CJ⊥GJ
1064. CJ⊥GK
1065. CJ⊥GL
1066. CJ⊥GM
1067. CJ⊥GN

1284. ABIHK
1285. ABIIL
1286. ABIJK
1287. ACIAI
1288. ACIAJ
1289. ACIBL
1290. ACICI
1291. ACICJ
1292. ACIDH
1293. ACIDK
1294. ACIEG
1295. ACIEM
1296. ACIEN
1297. ACIGM
1298. ACIGN
1299. ACIHK
1300. ACIIJ
1301. ACIMN
1302. ADIAI
1303. ADIBC
1304. ADIBJ
1305. ADIBL
1306. ADICJ
1307. ADICL
1308. ADIDI
1309. ADIEH
1310. ADIEK
1311. ADIFG
1312. ADIHK
1313. ADIJL
1314. AEIAI
1315. AEIBD
1316. AEIBL
1317. AEICJ
1318. AEIDL
1319. AEIEI
1320. AEIFH
1321. AEIFK
1322. AEIFM
1323. AEIHK
1324. AEIHM
1325. AEIKM
1326. AFIAI
1327. AFIBE
1328. AFIBL
1329. AFICD
1330. AFICJ
1331. AFIDJ

1548. BJICJ
1549. BJICL
1550. BJIDI
1551. BJIEH
1552. BJIEK
1553. BJIFG
1554. BJIHK
1555. BJIJL
1556. BKIBL
1557. BKICG
1558. BKICJ
1559. BKIDF
1560. BKIDN
1561. BKIFN
1562. BKIGJ
1563. BKIHK
1564. BKIHL
1565. BKIKL
1566. BLIBM
1567. BLIBN
1568. BLICD
1569. BLICE
1570. BLICF
1571. BLICG
1572. BLICH
1573. BLICI
1574. BLICJ
1575. BLICK
1576. BLICL
1577. BLICM
1578. BLICN
1579. BLIDE
1580. BLIDF
1581. BLIDG
1582. BLIDH
1583. BLIDI
1584. BLIDJ
1585. BLIDK
1586. BLIDL
1587. BLIDM
1588. BLIDN
1589. BLIEF
1590. BLIEG
1591. BLIEH
1592. BLIEI
1593. BLIEJ
1594. BLIEK
1595. BLIEL

1812. DNIKL
1813. EFIHJ
1814. EFIHK
1815. EFIIL
1816. EFIJK
1817. EGIEM
1818. EGIEN
1819. EGIGM
1820. EGIGN
1821. EGIHK
1822. EGIIJ
1823. EGIMN
1824. EHIEK
1825. EHIFG
1826. EHIHK
1827. EHIJL
1828. EIIFH
1829. EIIFK
1830. EIIFM
1831. EIIHK
1832. EIIHM
1833. EIIKM
1834. EJIFL
1835. EJIGI
1836. EJIHK
1837. EKIFG
1838. EKIHK
1839. EKIJL
1840. ELIFI
1841. ELIGH
1842. ELIGK
1843. ELIHK
1844. EMIEN
1845. EMIGM
1846. EMIGN
1847. EMIHK
1848. EMIIJ
1849. EMIMN
1850. ENIGM
1851. ENIGN
1852. ENIHK
1853. ENIIJ
1854. ENIMN
1855. FGIHK
1856. FGIJL
1857. FHIFK
1858. FHIFM
1859. FHIHK

2031. J∈○ADG
2032. K∈○ADG
2033. L∈○ADG
2034. I∈○ADH
2035. J∈○ADH
2036. K∈○ADH
2037. L∈○ADH
2038. J∈○ADI
2039. K∈○ADI
2040. L∈○ADI
2041. M∈○ADI
2042. N∈○ADI
2043. K∈○ADJ
2044. L∈○ADJ
2045. L∈○ADK
2046. G∈○AEF
2047. H∈○AEF
2048. I∈○AEF
2049. J∈○AEF
2050. K∈○AEF
2051. L∈○AEF
2052. H∈○AEG
2053. I∈○AEG
2054. J∈○AEG
2055. K∈○AEG
2056. L∈○AEG
2057. I∈○AEH
2058. J∈○AEH
2059. K∈○AEH
2060. L∈○AEH
2061. J∈○AEI
2062. K∈○AEI
2063. L∈○AEI
2064. M∈○AEI
2065. N∈○AEI
2066. K∈○AEJ
2067. L∈○AEJ
2068. L∈○AEK

2241. K∈○BHI
2242. L∈○BHI
2243. K∈○BHJ
2244. L∈○BHJ
2245. L∈○BHK
2246. M∈○BHK
2247. N∈○BHK
2248. M∈○BHL
2249. N∈○BHL
2250. K∈○BIJ
2251. L∈○BIJ
2252. L∈○BIK
2253. M∈○BIL
2254. N∈○BIL
2255. L∈○BJK
2256. M∈○BJL
2257. N∈○BJL
2258. M∈○BKL
2259. N∈○BKL
2260. F∈○CDE
2261. G∈○CDE
2262. H∈○CDE
2263. I∈○CDE
2264. J∈○CDE
2265. K∈○CDE
2266. L∈○CDE
2267. G∈○CDF
2268. H∈○CDF
2269. I∈○CDF
2270. J∈○CDF
2271. K∈○CDF
2272. L∈○CDF
2273. H∈○CDG
2274. I∈○CDG
2275. J∈○CDG

2454. N∈○EHK
2455. K∈○EIJ
2456. L∈○EIJ
2457. L∈○EIK
2458. L∈○EJK
2459. I∈○FGH
2460. J∈○FGH
2461. K∈○FGH
2462. L∈○FGH
2463. J∈○FGI
2464. K∈○FGI
2465. L∈○FGI
2466. K∈○FGJ
2467. L∈○FGJ
2468. L∈○FGK
2469. J∈○FHI
2470. K∈○FHI
2471. L∈○FHI
2472. K∈○FHJ
2473. L∈○FHJ
2474. L∈○FHK
2475. M∈○FHK
2476. N∈○FHK
2477. K∈○FIJ
2478. L∈○FIJ
2479. L∈○FIK
2480. L∈○FJK
2481. J∈○GHI
2482. K∈○GHI
2483. L∈○GHI
2484. K∈○GHJ
2485. L∈○GHJ
2486. L∈○GHK
2487. M∈○GHK
2488. N∈○GHK
2489. K∈○GIJ
2490. L∈○GIJ
2491. L∈○GIK
2492. L∈○GJK

329. BK=FK
330. BK=GI
331. BK=HL
332. BK=IJ
333. BK=KL
334. BL=CJ
335. BL=HK
336. BM=BN
337. BM=CM
338. BM=IN
339. BM=JM
340. BM=LM
341. BM=LN
342. BN=CM
343. BN=IN
344. BN=JM
345. BN=LN
346. CD=CL
347. CD=DE
348. CD=DJ
349. CD=EF
350. CD=FG
351. CD=GH
352. CD=GK
353. CD=HI
354. CD=IK
355. CD=IL
356. CD=JL
357. CE=CI
358. CE=DF
359. CE=DL

669. AF⊥JK
670. AG⊥AI
671. AG⊥AJ
672. AG⊥BL
673. AG⊥CI
674. AG⊥CJ
675. AG⊥DH
676. AG⊥DK
677. AG⊥EG
678. AG⊥EM
679. AG⊥EN
680. AG⊥GM
681. AG⊥GN
682. AG⊥HK
683. AG⊥IJ
684. AG⊥MN
685. AH⊥AI
686. AH⊥BC
687. AH⊥BJ
688. AH⊥BL
689. AH⊥CJ
690. AH⊥CL
691. AH⊥DI
692. AH⊥EH
693. AH⊥EK
694. AH⊥FG
695.

988. CD⊥JK
989. CE⊥CI
990. CE⊥CJ
991. CE⊥DH
992. CE⊥DK
993. CE⊥EG
994. CE⊥EM
995. CE⊥EN
996. CE⊥GM
997. CE⊥GN
998. CE⊥HK
999. CE⊥IJ
1000. CE⊥MN
1001. CF⊥CJ
1002. CF⊥CL
1003. CF⊥DI
1004. CF⊥EH
1005. CF⊥EK
1006. CF⊥FG
1007. CF⊥HK
1008. CF⊥JL
1009. CG⊥CJ
1010. CG⊥DL
1011. CG⊥EI
1012.

1202. FH⊥GJ
1203. FH⊥HK
1204. FH⊥HL
1205. FH⊥KL
1206. FI⊥HJ
1207. FI⊥HK
1208. FI⊥IL
1209. FI⊥JK
1210. FJ⊥HK
1211. FJ⊥JL
1212. FK⊥FN
1213. FK⊥GJ
1214. FK⊥HK
1215. FK⊥HL
1216. FK⊥KL
1217. FL⊥GM
1218. FL⊥GN
1219. FL⊥HK
1220. FL⊥IJ
1221.

1447. AJ∥EN
1448. AJ∥GM
1449. AJ∥GN
1450. AJ∥HK
1451. AJ∥IJ
1452. AK∥BG
1453. AK∥BL
1454. AK∥CF
1455. AK∥CJ
1456. AK∥DE
1457. AK∥FJ
1458. AK∥GL
1459. AK∥HI
1460. AK∥HK
1461. AK∥IK
1462. AL∥BI
1463. AL∥BL
1464. AL∥CH
1465. AL∥CJ
1466. AL∥CK
1467. AL∥DG
1468. AL∥EF
1469. AL∥HJ
1470.

1706. CJ∥EN
1707. CJ∥FG
1708. CJ∥FH
1709. CJ∥FI
1710. CJ∥FJ
1711. CJ∥FK
1712. CJ∥FL
1713. CJ∥FM
1714. CJ∥FN
1715. CJ∥GH
1716. CJ∥GI
1717. CJ∥GJ
1718. CJ∥GK
1719. CJ∥GL
1720. CJ∥GM
1721. CJ∥GN
1722. CJ∥HI
1723. CJ∥HJ
1724. CJ∥HK
1725. CJ∥HL
1726. CJ∥HM
1727. CJ∥HN
1728. CJ∥IJ
1729. CJ∥IK
1730. CJ∥IL
1731. CJ∥IM
1732. CJ∥IN
1733. CJ∥JK
1734. CJ∥JL
1735. CJ∥JM
1736. CJ∥JN

1948. K∈○ABE
1949. L∈○ABE
1950. G∈○ABF
1951. H∈○ABF
1952. I∈○ABF
1953. J∈○ABF
1954. K∈○ABF
1955. L∈○ABF
1956. H∈○ABG
1957. I∈○ABG
1958. J∈○ABG
1959. K∈○ABG
1960. L∈○ABG
1961. I∈○ABH
1962. J∈○ABH
1963. K∈○ABH
1964. L∈○ABH

2128. H∈○BCE
2129. I∈○BCE
2130. J∈○BCE
2131. K∈○BCE
2132. L∈○BCE
2133. G∈○BCF
2134. H∈○BCF
2135. I∈○BCF
2136. J∈○BCF
2137. K∈○BCF
2138. L∈○BCF
2139. H∈○BCG
2140. I∈○BCG
2141. J∈○BCG
2142. K∈○BCG
2143. L∈○BCG
2144. I∈○BCH

2308. K∈○CEJ
2309. L∈○CEJ
2310. M∈○CEJ
2311. N∈○CEJ
2312. L∈○CEK
2313. H∈○CFG
2314. I∈○CFG
2315. J∈○CFG
2316. K∈○CFG
2317. L∈○CFG
2318. I∈○CFH
2319. J∈○CFH
2320. K∈○CFH
2321. L∈○CFH
2322. J∈○CFI
2323. K∈○CFI
2324. L∈○CFI

2488. N∈○GHK
2489. K∈○GIJ
2490. L∈○GIJ
2491. L∈○GIK
2492. L∈○GJK
2493. K∈○HIJ
2494. L∈○HIJ
2495. L∈○HIK
2496. M∈○HIK
2497. N∈○HIK
2498. L∈○HJK
2499. M∈○HJK
2500. N∈○HJK
2501. M∈○HKL
2502. N∈○HKL
2503. L∈○IJK
2504. N∈○IJM

Finished, found 2504 theorems among 13650 possible statements.
Found theorems that are true only on parts.
Elapsed time: 0h 8m 18s

Questions:

Is there a better algorithm than this one?

Do we need to compute full Gbasis for deciding if Elimination is zero or if Ideal is 1?

Monomial order dependence?

One that guarantees computing the lowest degree?

Adding non-degeneracy conditions, how to grade their discovery?

- Is there a relation of degree of syzygies and complexity of membership problem (for example, if thesis is in the ideal of hypotheses)?

- Is there a relation between the degree of syzygyies in the reductio ad absurdum and the minimal power of the thesis in the radical of hypotheses?

Over the reals:

Generally true: t=0 holds over irreducible components of V(h1, …, hr) over the reals, where {free} are independent.

Projection(V(h1,…,hr, t*z-1) over $\mathbb{R}^{\{free\}} \sqsubseteq$ Hypersurface

If n belongs to this elimination ideal, then

1 belongs to the real radical <h1,…,hr, t*z-1, n*w-1>

And showing 1 as a combination of {h1,…,hr, t*z-1, n*w-1} is a proof certificate , complexity measure.

REAL Questions

Is there a specific algorithm to decide if a projection is proper?

Is there an (implemented) algorithm to decide if 1 is in the real radical of an ideal I=<f_1, ...,fr>, and to compute the “simplest” Nullstellensatz-like expression (degree of SOS, degree of p_i)

$$1+ SOS=p1*f1+...+pr*fr$$

Proving/involving inequalities:

Given $H:=\{f_1=0,\ldots,f_r=0, g_1>0, \ldots., g_s>0\}$

$T = 0$ over H iff....

$T \geq 0$ over H iff....

$T > 0$ over H iff....

Expression, algorithm, degree of involved expressions

Currrent work:

benchmark /comparison difficulty of humans/ AI bots

extending method to inequalities

solving many involved computer algebra problems related to our approach

- We have yet to perform a large number of tests (with different statements: labeled as old, new, basic, advanced, etc.) to get some "human" perception about the meaning of our measure.

- Our approach has launched a collection of seemingly open, basic, mathematical questions in Computational Algebraic Geometry (after consultation with experts).

- This is initial work.

- Ariño-Morera, B., Kovács, Z., Recio, T., and Tolmos, P.: “Solving with GeoGebra Discovery an Austrian Mathematics Olympiad problem: Lessons Learned”. In: Quaresma, P. and Kovács, Z. (eds), Proceedings 14th International Conference on Automated Deduction in Geometry, Belgrade, Serbia, 20-22th September 2023, Electronic Proceedings in Theoretical Computer Science 398, Open Publishing Association, pp. 101-109, January 2024. doi: 10.4204/EPTCS.398.13

- Kovács, Z., Recio, T., Vélez, M. P.: “Showing Proofs, Assessing Difficulty with GeoGebra Discovery”. In: Quaresma, P. and Kovács, Z. (eds), Proceedings 14th International Conference on Automated Deduction in Geometry, Belgrade, Serbia, 20-22th September 2023, Electronic Proceedings in Theoretical Computer Science 398, Open Publishing Association, pp. 43-52, January 2024. doi: 10.4204/EPTCS.398.8

THANKS!